

Homework 5

Problem 1 (4pts). Let $L \subseteq \mathbb{R}^n$ be a linear subspace, and let P_L denote the orthogonal projection onto L . Prove the following properties:

1. $P_L(x) \in L$ for every $x \in \mathbb{R}^n$.
2. $x - P_L(x) \in L^\perp$ for every $x \in \mathbb{R}^n$.
3. If $x \in L$, then

$$P_L(x) = x.$$

4. P_L is linear.
5. P_L is idempotent; that is,

$$P_L(P_L(x)) = P_L(x)$$

for every $x \in \mathbb{R}^n$.

6. Prove the Pythagorean identity

$$\|x\|_2^2 = \|P_L(x)\|_2^2 + \|x - P_L(x)\|_2^2.$$

Problem 2 (6pts). Let

$$U = \{u_\ell\}_{\ell=1}^r \quad \text{and} \quad V = \{v_\ell\}_{\ell=1}^r$$

be two orthonormal bases of the same linear subspace

$$L = \text{span}(U) = \text{span}(V) \subseteq \mathbb{R}^n.$$

- (a) Show that for every $x \in \mathbb{R}^n$ and every $j = 1, \dots, r$,

$$\langle u_j, x \rangle = \sum_{\ell=1}^r \langle v_\ell, x \rangle \langle u_j, v_\ell \rangle.$$

- (b) Define

$$P_U(x) = \sum_{j=1}^r \langle u_j, x \rangle u_j \quad \text{and} \quad P_V(x) = \sum_{\ell=1}^r \langle v_\ell, x \rangle v_\ell.$$

Use part (a) to show that

$$P_U(x) = P_V(x) \quad \text{for every } x \in \mathbb{R}^n.$$

Thus, the formula for the orthogonal projection onto L is independent of the choice of orthonormal basis of L .