

Homework 4 Solutions

Problem 1.

Let

$$L = \text{Col}(Q).$$

Since the columns of Q form an orthonormal basis for L ,

$$P_L = QQ^T.$$

Therefore,

$$(I_m - QQ^T)x = (I_m - P_L)x = P_{L^\perp}x \in L^\perp.$$

On the other hand,

$$QQ^Ty = P_Ly \in L.$$

Hence,

$$(I_m - QQ^T)x \perp QQ^Ty.$$

Problem 2.

Let the columns of U be

$$u_1, \dots, u_n.$$

Since U is orthogonal, its columns form an orthonormal basis, so

$$\|u_j\|_2 = 1, \quad j = 1, \dots, n.$$

By the definition of the Frobenius norm,

$$\|U\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n |U_{ij}|^2.$$

Grouping the entries by columns gives

$$\|U\|_F^2 = \sum_{j=1}^n \|u_j\|_2^2.$$

Since each column has unit norm,

$$\|U\|_F^2 = \sum_{j=1}^n 1 = n.$$

Therefore,

$$\boxed{\|U\|_F = \sqrt{n}.$$

Problem 3.

Recall the dimension formula

$$\dim(L + K) = \dim(L) + \dim(K) - \dim(L \cap K).$$

Since

$$L + K \subseteq \mathbb{R}^n,$$

we have

$$\dim(L + K) \leq n.$$

Therefore,

$$\dim(L) + \dim(K) - \dim(L \cap K) \leq n,$$

so

$$\dim(L \cap K) \geq \dim(L) + \dim(K) - n.$$

By assumption,

$$\dim(L) + \dim(K) > n.$$

Since all dimensions are integers,

$$\dim(L) + \dim(K) - n \geq 1.$$

Hence,

$$\boxed{\dim(L \cap K) \geq 1.}$$

Problem 4.

Let the columns of A be

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Notice that

$$a_2 = 2a_1, \quad a_3 = 3a_1.$$

Thus, the first three columns span only one direction.

We have

$$\|a_1\|_2 = 2,$$

so

$$q_1 = \frac{a_1}{\|a_1\|_2} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Also,

$$q_1^T a_4 = \frac{1}{2}(1 - 1 + 1 - 1) = 0.$$

Thus a_4 is already orthogonal to q_1 . Since

$$\|a_4\|_2 = 2,$$

we obtain

$$q_2 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Therefore, a reduced QR decomposition is

$$A = QR,$$

where

$$Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$R = Q^T A,$$

we obtain

$$R = \begin{pmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hence,

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Notice that

$$\text{rank}(A) = 2,$$

so the reduced Q has two columns and the reduced R has two rows.