

## Solutions

**Problem 1.** Let  $S, T \subseteq \mathbb{R}^n$ . We want to show that

$$\text{span}(T \cap S) \subseteq \text{span}(T) \cap \text{span}(S).$$

Let

$$x \in \text{span}(T \cap S).$$

Then there exist  $v_1, \dots, v_k \in T \cap S$  and scalars  $c_1, \dots, c_k$  such that

$$x = c_1 v_1 + \dots + c_k v_k.$$

Since each  $v_i \in T \cap S$ , we have

$$v_i \in T \quad \text{and} \quad v_i \in S.$$

Therefore,

$$x \in \text{span}(T) \quad \text{and} \quad x \in \text{span}(S).$$

Hence

$$x \in \text{span}(T) \cap \text{span}(S).$$

Thus,

$$\boxed{\text{span}(T \cap S) \subseteq \text{span}(T) \cap \text{span}(S)}.$$

**Problem 2.** Let  $A \in \mathbb{R}^{m \times n}$ . Recall that

$$\text{rank}(A) = \dim(\text{Col}(A)).$$

Since  $A$  has  $n$  columns, its column space is spanned by at most  $n$  vectors. Therefore,

$$\text{rank}(A) \leq n.$$

On the other hand,

$$\text{Col}(A) \subseteq \mathbb{R}^m.$$

Since every subspace of  $\mathbb{R}^m$  has dimension at most  $m$ ,

$$\text{rank}(A) \leq m.$$

Combining these two inequalities gives

$$\boxed{\text{rank}(A) \leq \min\{m, n\}}.$$

**Problem 3.** Let

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

We apply the Gram–Schmidt process.

First,

$$u_1 = v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Since

$$\|u_1\| = \sqrt{1+1+1} = \sqrt{3},$$

we obtain

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Next,

$$u_2 = v_2 - \text{proj}_{u_1}(v_2).$$

Since

$$v_2^T u_1 = 1 - 1 + 0 = 0,$$

$v_1$  and  $v_2$  are already orthogonal. Thus

$$u_2 = v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Since

$$\|u_2\| = \sqrt{2},$$

we obtain

$$q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

For the third vector,

$$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3).$$

We have

$$v_3^T u_1 = 4 + 0 + 2 = 6, \quad u_1^T u_1 = 3,$$

so

$$\text{proj}_{u_1}(v_3) = \frac{6}{3} u_1 = 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

Also,

$$v_3^T u_2 = 4, \quad u_2^T u_2 = 2,$$

so

$$\text{proj}_{u_2}(v_3) = \frac{4}{2} u_2 = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

Therefore,

$$u_3 = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus  $v_3$  is already in the span of  $v_1$  and  $v_2$ . In fact,

$$v_3 = 2v_1 + 2v_2.$$

Therefore, the Gram–Schmidt process produces the basis

$$B = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

To verify that  $B$  is orthonormal, first note that

$$\|q_1\| = \|q_2\| = 1.$$

Moreover,

$$q_1^T q_2 = \frac{1}{\sqrt{6}}(1 - 1 + 0) = 0.$$

Hence the vectors are orthogonal and both have unit norm.

Since Gram–Schmidt does not change the span,

$$\text{span}(B) = \text{span}(S).$$

Therefore,

$$B \text{ is an orthonormal basis for } \text{span}(S).$$