

Lecture 9: Review and Challenge Problems

Theorem 1. *Any two bases of a finite-dimensional subspace $V \subseteq \mathbb{C}^n$ have the same number of vectors.*

Proof. Let

$$B_1 = \{u_1, \dots, u_m\}, \quad B_2 = \{v_1, \dots, v_k\}$$

be two bases of V .

Apply Gram–Schmidt to B_1 and B_2 . Since both sets are linearly independent, Gram–Schmidt does not remove any vectors. Thus we obtain two orthonormal bases

$$Q = \{q_1, \dots, q_m\}, \quad P = \{p_1, \dots, p_k\}$$

of V .

Since P is an orthonormal basis of V , for every q_i ,

$$q_i = \sum_{j=1}^k \langle p_j, q_i \rangle p_j.$$

Hence, by orthonormality,

$$1 = \|q_i\|^2 = \sum_{j=1}^k |\langle p_j, q_i \rangle|^2.$$

Summing over i gives

$$m = \sum_{i=1}^m \sum_{j=1}^k |\langle p_j, q_i \rangle|^2.$$

Similarly, since Q is an orthonormal basis of V ,

$$1 = \sum_{i=1}^m |\langle q_i, p_j \rangle|^2$$

for every p_j . Therefore,

$$k = \sum_{j=1}^k \sum_{i=1}^m |\langle q_i, p_j \rangle|^2.$$

Since

$$|\langle p_j, q_i \rangle| = |\langle q_i, p_j \rangle|,$$

the two double sums are equal. Hence

$$m = k.$$

□

Lemma 2. Let $\mathcal{L}, \mathcal{K} \subseteq \mathbb{C}^n$ be two linear subspaces such that

$$\mathcal{L} \cap \mathcal{K} = \{0\}.$$

Then $\mathcal{L} \cup \mathcal{K}$ contains

$$\dim(\mathcal{L}) + \dim(\mathcal{K})$$

linearly independent vectors.

Proof. Let

$$r = \dim(\mathcal{L}), \quad s = \dim(\mathcal{K}),$$

and let

$$B = \{b_1, \dots, b_r\}$$

be a basis of $\mathcal{L}$ and

$$A = \{a_1, \dots, a_s\}$$

be a basis of $\mathcal{K}$.

Since

$$\mathcal{L} \cap \mathcal{K} = \{0\},$$

and neither A nor B contains the zero vector, the set

$$B \cup A$$

contains exactly $r + s$ vectors.

We claim that $B \cup A$ is linearly independent. Suppose

$$\sum_{j=1}^r \alpha_j b_j + \sum_{k=1}^s \beta_k a_k = 0.$$

Then

$$\sum_{j=1}^r \alpha_j b_j = - \sum_{k=1}^s \beta_k a_k.$$

The left-hand side belongs to $\mathcal{L}$, while the right-hand side belongs to $\mathcal{K}$. Hence their common value belongs to

$$\mathcal{L} \cap \mathcal{K} = \{0\}.$$

Therefore,

$$\sum_{j=1}^r \alpha_j b_j = 0, \quad \sum_{k=1}^s \beta_k a_k = 0.$$

Since B and A are bases, they are linearly independent. Thus

$$\alpha_1 = \dots = \alpha_r = 0, \quad \beta_1 = \dots = \beta_s = 0.$$

Hence $B \cup A$ is linearly independent and contains

$$r + s = \dim(\mathcal{L}) + \dim(\mathcal{K})$$

vectors. □

Theorem 3. Let $B \subset \mathbb{R}^n$ be an orthonormal basis of a linear subspace $L \subset \mathbb{R}^n$. Then there exists an orthonormal basis $\tilde{B}$ of $\mathbb{R}^n$ such that

$$B \subset \tilde{B}.$$

Proof. Suppose

$$B = \{q_1, \dots, q_k\}.$$

Since B is an orthonormal set, it is linearly independent. Therefore, we can extend B to a basis of $\mathbb{R}^n$:

$$\{q_1, \dots, q_k, v_{k+1}, \dots, v_n\}.$$

Now apply the Gram–Schmidt process to these vectors in this order.

Since $q_1, \dots, q_k$ are already orthonormal, the Gram–Schmidt process leaves them unchanged. Thus, the first k vectors produced are still

$$q_1, \dots, q_k.$$

Applying Gram–Schmidt to the remaining vectors $v_{k+1}, \dots, v_n$ produces vectors

$$\tilde{q}_{k+1}, \dots, \tilde{q}_n$$

such that

$$\tilde{B} = \{q_1, \dots, q_k, \tilde{q}_{k+1}, \dots, \tilde{q}_n\}$$

is an orthonormal basis of $\mathbb{R}^n$.

Therefore,

$$B \subseteq \tilde{B}.$$

□

Definition (Sum of Subspaces). Let $U, V \subseteq \mathbb{R}^m$ be subspaces. The sum of U and V is defined by

$$U + V = \{u + v : u \in U, v \in V\}.$$

Lemma 4. Let $U, V \subseteq \mathbb{R}^m$ be subspaces. Then

$$\dim(U + V) \leq \dim(U) + \dim(V).$$

Proof. Let

$$\{u_1, \dots, u_r\}$$

be a basis of U , and let

$$\{v_1, \dots, v_s\}$$

be a basis of V .

We claim that

$$\{u_1, \dots, u_r, v_1, \dots, v_s\}$$

spans $U + V$.

Indeed, for any $x \in U + V$, by definition of the sum of subspaces, there exist $u \in U$ and $v \in V$ such that

$$x = u + v.$$

Since $\{u_1, \dots, u_r\}$ is a basis of U and $\{v_1, \dots, v_s\}$ is a basis of V , we can write

$$u = \sum_{i=1}^r a_i u_i, \quad v = \sum_{j=1}^s b_j v_j.$$

Therefore,

$$x = \sum_{i=1}^r a_i u_i + \sum_{j=1}^s b_j v_j.$$

Hence,

$$U + V = \text{span}\{u_1, \dots, u_r, v_1, \dots, v_s\}.$$

Thus $U + V$ is spanned by $r + s$ vectors, so

$$\dim(U + V) \leq r + s = \dim(U) + \dim(V).$$

□