Lecture 15: The Spectral Theorem for Hermitian Matrices

We now study a particularly important class of square matrices: Hermitian matrices. They will play a central role when we later introduce the singular value decomposition (SVD).

Definition (Hermitian Matrix)**.** *For a complex matrix A, let*

$$A^* = \overline{A}^T$$

denote its conjugate transpose*.*

A matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian *if*

$$A = A^*.$$

An important source of Hermitian matrices is provided by matrices of the form A^*A and AA^*.

Proposition 0.1. *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$A^*A \in \mathbb{C}^{n\times n} \qquad \text{and} \qquad AA^* \in \mathbb{C}^{m\times m}$$

are Hermitian.

Proof. Using $(BC)^* = C^*B^*$, we have

$$(A^*A)^* = A^*A$$

and

$$(AA^*)^* = AA^*.$$

□

1 Eigenvalues and Eigenvectors

Definition (Eigenvalue and Eigenvector)**.** *Let $A \in \mathbb{C}^{n\times n}$. A scalar λ and a nonzero vector $v \in \mathbb{C}^n$ form an* eigenvalue–eigenvector pair *if*

$$Av = \lambda v.$$

The scalar λ is called an eigenvalue *of A, and v is called an* eigenvector *associated with λ.*

The equation

$$Av = \lambda v$$

can equivalently be written as

$$(A - \lambda I_n)v = 0.$$

Thus, the eigenvectors associated with λ, together with the zero vector, form the subspace

$$\mathcal{N}(A - \lambda I_n).$$

Definition (Eigenspace)**.** *The* eigenspace *associated with an eigenvalue λ is*

$$E_\lambda = \mathcal{N}(A - \lambda I_n).$$

Notice that every nonzero vector in E_λ is an eigenvector associated with λ.

Computing Eigenvalues and Eigenvectors

Let

$$A \in \mathbb{C}^{n \times n}.$$

To find the eigenvalues of A, recall that

$$Av = \lambda v$$

is equivalent to

$$(A - \lambda I)v = 0.$$

For a nonzero solution v to exist, the matrix

$$A - \lambda I$$

must be singular. Therefore,

$$\boxed{\det(A - \lambda I) = 0.}$$

This equation is called the *characteristic equation.*

Its solutions are the eigenvalues of A.

Once an eigenvalue λ is known, the corresponding eigenvectors are the nonzero solutions of

$$(A - \lambda I)v = 0.$$

Equivalently, the eigenspace is

$$E_\lambda = \mathcal{N}(A - \lambda I).$$

Example 1.1. *Let*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

To find the eigenvalues, solve

$$\det(A - \lambda I) = 0.$$

We have

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix}.$$

Therefore,

$$\det(A - \lambda I) = (2 - \lambda)^2 - 1.$$

Thus,

$$(2 - \lambda)^2 - 1 = 0,$$

so

$$\lambda = 3 \qquad \textit{or} \qquad \lambda = 1.$$

For $\lambda = 3$,

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solving

$$(A - 3I)v = 0$$

gives

$$v_1 = v_2.$$

Hence,

$$E_3 = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

For $\lambda = 1$,

$$A - I = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}.$$

Solving

$$(A - I)v = 0$$

gives

$$v_1 = -v_2.$$

Hence,

$$E_1 = \operatorname{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}.$$

After normalization,

$$q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Eigenvalues of Hermitian Matrices are Real

Hermitian matrices have an important property that does not hold for general complex matrices: all of their eigenvalues are real.

Theorem 1.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Then every eigenvalue of A is real.*

Proof. Suppose

$$Av = \lambda v, \qquad v \neq 0.$$

Then

$$v^*Av = \lambda v^*v = \lambda\|v\|_2^2.$$

Because $A = A^*$,

$$\overline{v^*Av} = v^*A^*v = v^*Av,$$

so v^*Av is real. Since $\|v\|_2^2 > 0$,

$$\lambda = \frac{v^*Av}{\|v\|_2^2} \in \mathbb{R}.$$

□

Orthogonality of Eigenvectors

Hermitian matrices have a remarkable geometric property: eigenvectors corresponding to different eigenvalues are orthogonal.

Theorem 1.3. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Suppose*

$$Av = \lambda v, \qquad Au = \mu u,$$

with $\lambda \neq \mu$. Then

$$v \perp u.$$

Proof. By the previous theorem, $\lambda, \mu \in \mathbb{R}$. Since $A = A^*$,

$$\lambda v^* u = (Av)^* u = v^* A^* u = v^* A u = \mu v^* u.$$

Therefore,

$$(\lambda - \mu) v^* u = 0.$$

Since $\lambda \neq \mu$,

$$\boxed{v^* u = 0.}$$

Hence $v \perp u$. □

More generally, eigenspaces associated with distinct eigenvalues are orthogonal:

$$E_\lambda \perp E_\mu \qquad \text{whenever } \lambda \neq \mu.$$

2 The Spectral Theorem

The previous theorem tells us that eigenvectors associated with different eigenvalues are automatically orthogonal.

The much deeper fact is that a Hermitian matrix has enough eigenvectors to form a basis of all of $\mathbb{C}^n$.

Theorem 2.1 (Spectral Theorem)**.** *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Then there exists an orthonormal basis*

$$\{q_1, \ldots, q_n\}$$

of $\mathbb{C}^n$ consisting entirely of eigenvectors of A.

We will use this fundamental theorem without proving the existence of the eigenbasis.

Suppose

$$Aq_j = \lambda_j q_j, \qquad j = 1, \ldots, n,$$

and let

$$Q = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix}.$$

Because the q_j form an orthonormal basis,

$$Q^*Q = QQ^* = I_n,$$

so Q is a unitary matrix.

Now observe that

$$AQ = \begin{pmatrix} | & | & & | \\ Aq_1 & Aq_2 & \cdots & Aq_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1 q_1 & \lambda_2 q_2 & \cdots & \lambda_n q_n \\ | & | & & | \end{pmatrix}.$$

Therefore,

$$AQ = Q \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Let

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Then

$$AQ = Q\Lambda.$$

Multiplying on the right by Q^* gives

$$\boxed{A = Q\Lambda Q^*.}$$

This is called the *spectral decomposition* of A.

Proposition 2.2 (Spectral Decomposition)**.** *A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if there exists a unitary matrix Q and a real diagonal matrix Λ such that*

$$\boxed{A = Q\Lambda Q^*.}$$

Proof. The forward direction is the spectral theorem.

For the reverse direction, suppose

$$A = Q\Lambda Q^*,$$

where Λ is real and diagonal. Then

$$A^* = (Q\Lambda Q^*)^* = Q\Lambda^* Q^* = Q\Lambda Q^* = A.$$

Thus A is Hermitian. □

Geometric Interpretation

The decomposition

$$A = Q\Lambda Q^*$$

gives a simple geometric interpretation of a Hermitian linear map.

For any $x \in \mathbb{C}^n$,

$$Ax = Q\Lambda Q^* x.$$

Thus the transformation can be viewed in three steps:

1. Compute
$$z = Q^* x.$$
This expresses x in the orthonormal eigenvector basis.

2. Apply
$$\Lambda z.$$
Each coordinate is scaled independently by its corresponding eigenvalue.

3. Multiply by Q to return to the original coordinates.

Thus Hermitian matrices have especially simple behavior when viewed in the correct orthonormal coordinate system.

Spectral Decomposition as a Sum of Rank-One Matrices

Let
$$Q = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix}.$$

Then
$$A = Q\Lambda Q^*$$
can also be written as
$$\boxed{A = \sum_{j=1}^{n} \lambda_j q_j q_j^*.}$$

Indeed,
$$Q\Lambda Q^* = \sum_{j=1}^{n} \lambda_j q_j q_j^*.$$

Each matrix
$$q_j q_j^*$$
is the orthogonal projection onto
$$\text{span}\{q_j\}.$$

Thus a Hermitian matrix can be viewed as a weighted sum of orthogonal projections onto its eigendirections.

Zero Eigenvalues and the Null Space

The eigenvalue 0 has a particularly simple interpretation.

Indeed,
$$Av = 0v$$

if and only if

$$Av = 0.$$

Therefore,

$$\boxed{E_0 = \mathcal{N}(A).}$$

Lemma 2.3. *Let A be invertible. Then for any matrix B of compatible size,*

$$\operatorname{rank}(AB) = \operatorname{rank}(B).$$

Similarly,

$$\operatorname{rank}(BA) = \operatorname{rank}(B).$$

Proof. Suppose the columns of B are

$$b_1, \ldots, b_n.$$

Then the columns of AB are

$$Ab_1, \ldots, Ab_n.$$

We show that the same linear dependence relations hold among these two sets of columns.

Indeed,

$$c_1 Ab_1 + \cdots + c_n Ab_n = 0$$

if and only if

$$A(c_1 b_1 + \cdots + c_n b_n) = 0.$$

Since A is invertible,

$$Az = 0$$

implies

$$z = 0.$$

Therefore,

$$c_1 Ab_1 + \cdots + c_n Ab_n = 0$$

if and only if

$$c_1 b_1 + \cdots + c_n b_n = 0.$$

Hence the columns of AB have exactly the same linear dependence relations as the columns of B.

Therefore,

$$\boxed{\operatorname{rank}(AB) = \operatorname{rank}(B).}$$

□

This immediately relates the rank of a Hermitian matrix to its nonzero eigenvalues.

Suppose

$$A = Q\Lambda Q^*.$$

Because Q and Q^* are invertible,

$$\operatorname{rank}(A) = \operatorname{rank}(\Lambda).$$

But the rank of a diagonal matrix is simply the number of nonzero diagonal entries. Hence,

Theorem 2.4. *For a Hermitian matrix A,*

$$\boxed{\operatorname{rank}(A) = \textit{number of nonzero eigenvalues of } A,}$$

where eigenvalues are counted with multiplicity.

Compact Spectral Decomposition

Suppose

$$A \in \mathbb{C}^{n\times n}$$

is Hermitian and has rank $r < n$.

Then exactly r of its eigenvalues are nonzero.

Reorder the eigenvalues so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0,$$

while

$$\lambda_{r+1} = \cdots = \lambda_n = 0.$$

Let

$$Q_r = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_r \\ | & & | \end{pmatrix} \in \mathbb{C}^{n\times r},$$

and let

$$\Lambda_r = \operatorname{diag}(\lambda_1, \ldots, \lambda_r).$$

Then

$$\boxed{A = Q_r \Lambda_r Q_r^*.}$$

Theorem 2.5 (Compact Spectral Decomposition)**.** *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with rank r. Then there exists a matrix with orthonormal columns*

$$Q_r \in \mathbb{C}^{n\times r}$$

and a diagonal matrix

$$\Lambda_r \in \mathbb{C}^{r\times r}$$

with nonzero diagonal entries such that

$$\boxed{A = Q_r \Lambda_r Q_r^*.}$$

This representation stores only the eigendirections that actually contribute to A.

The Special Matrices A^*A and AA^*

Recall that for every matrix

$$A \in \mathbb{C}^{m \times n},$$

both

$$A^*A \qquad \text{and} \qquad AA^*$$

are Hermitian.

They have an additional important property: their eigenvalues are nonnegative.

Theorem 2.6. *Every eigenvalue of A^*A is nonnegative. Similarly, every eigenvalue of AA^* is nonnegative.*

Proof. Suppose

$$A^*Av = \lambda v$$

for some nonzero v.

Multiplying by v^* gives

$$v^*A^*Av = \lambda v^*v.$$

But

$$v^*A^*Av = (Av)^*(Av) = \|Av\|_2^2.$$

Therefore,

$$\lambda \|v\|_2^2 = \|Av\|_2^2 \geq 0.$$

Since $v \neq 0$,

$$\|v\|_2^2 > 0,$$

and hence

$$\boxed{\lambda \geq 0.}$$

The proof for AA^* is identical. □

This fact will become extremely important when we define the singular values of a matrix.

Example of a Spectral Decomposition

Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We have

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

After normalization,

$$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\boxed{A = Q\Lambda Q^*.}$$

Equivalently,

$$A = 3q_1 q_1^* + q_2 q_2^*.$$