

Lecture 11: Sums of Subspaces and beyond

1 Rank of a Sum

Theorem 1.1. *Let*

$$A, B \in \mathbb{R}^{m \times n}.$$

Then

$$\boxed{\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)}.$$

Proof

For any $x \in \mathbb{R}^n$,

$$(A + B)x = Ax + Bx.$$

Since

$$Ax \in \mathcal{C}(A)$$

and

$$Bx \in \mathcal{C}(B),$$

we have

$$(A + B)x \in \mathcal{C}(A) + \mathcal{C}(B).$$

Therefore,

$$\mathcal{C}(A + B) \subseteq \mathcal{C}(A) + \mathcal{C}(B).$$

Hence

$$\text{rank}(A + B) = \dim \mathcal{C}(A + B)$$

satisfies

$$\text{rank}(A + B) \leq \dim(\mathcal{C}(A) + \mathcal{C}(B)).$$

Using the dimension inequality,

$$\dim(\mathcal{C}(A) + \mathcal{C}(B)) \leq \dim \mathcal{C}(A) + \dim \mathcal{C}(B).$$

Therefore,

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

□

—

2 Orthogonal Subspaces

Definition. Two subspaces $U, V \subseteq \mathbb{R}^n$ are called ***orthogonal*** if

$$\langle u, v \rangle = 0$$

for every $u \in U$ and $v \in V$.

We write

$$U \perp V.$$

Orthogonal subspaces have no nonzero vector in common.

Proposition

If $U \perp V$, then

$$U \cap V = \{0\}.$$

Proof

Let $x \in U \cap V$. Since $x \in U$ and $x \in V$, orthogonality gives

$$\langle x, x \rangle = 0.$$

Thus

$$\|x\|_2^2 = 0,$$

which implies

$$x = 0.$$

Therefore

$$U \cap V = \{0\}.$$

□

Consequently,

$$U \perp V \implies U + V = U \oplus V$$

and

$$\dim(U + V) = \dim(U) + \dim(V).$$

—

3 Combining Orthonormal Bases

Theorem 3.1. *Suppose $U \perp V$.*

Let

$$\mathcal{B}_U = \{u_1, \dots, u_r\}$$

be an orthonormal basis of U , and let

$$\mathcal{B}_V = \{v_1, \dots, v_s\}$$

be an orthonormal basis of V .

Then

$$\mathcal{B}_U \cup \mathcal{B}_V$$

is an orthonormal basis of $U + V$.

Proof

Within $\mathcal{B}_U$,

$$\langle u_i, u_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Similarly, within $\mathcal{B}_V$,

$$\langle v_i, v_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Finally, because $U \perp V$,

$$\langle u_i, v_j \rangle = 0$$

for every i, j .

Hence

$$\mathcal{B}_U \cup \mathcal{B}_V$$

is an orthonormal set.

It remains to show that it spans $U + V$.

Take $x \in U + V$. Then

$$x = u + v$$

for some $u \in U$ and $v \in V$.

Since $\mathcal{B}_U$ and $\mathcal{B}_V$ are bases,

$$u = \sum_{i=1}^r a_i u_i, \quad v = \sum_{j=1}^s b_j v_j.$$

Thus

$$x = \sum_{i=1}^r a_i u_i + \sum_{j=1}^s b_j v_j,$$

so

$$x \in \text{span}(\mathcal{B}_U \cup \mathcal{B}_V).$$

Therefore

$$\mathcal{B}_U \cup \mathcal{B}_V$$

is an orthonormal basis of $U + V$. $\square$

—

4 Orthogonal Complements

Definition. Let $L \subseteq \mathbb{R}^n$ be a linear subspace. The orthogonal complement of L is

$$L^\perp = \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \text{ for every } y \in L\}.$$

Thus $L^\perp$ consists of all vectors perpendicular to the entire subspace L .

Theorem 4.1. $L^\perp$ is a linear subspace of $\mathbb{R}^n$.

Proof

Clearly,

$$\langle 0, y \rangle = 0$$

for every $y \in L$, so

$$0 \in L^\perp.$$

Now let $x_1, x_2 \in L^\perp$ and let $\alpha, \beta \in \mathbb{R}$.

For every $y \in L$,

$$\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle.$$

Since $x_1, x_2 \in L^\perp$,

$$\langle x_1, y \rangle = \langle x_2, y \rangle = 0.$$

Hence

$$\langle \alpha x_1 + \beta x_2, y \rangle = 0.$$

Therefore

$$\alpha x_1 + \beta x_2 \in L^\perp.$$

Thus $L^\perp$ is a linear subspace. $\square$

—

Dimension of the Orthogonal Complement

Let

$$\mathcal{B} = \{q_1, \dots, q_r\}$$

be an orthonormal basis of L .

By extending an orthonormal basis, we can find vectors

$$q_{r+1}, \dots, q_n$$

such that

$$\{q_1, \dots, q_r, q_{r+1}, \dots, q_n\}$$

is an orthonormal basis of $\mathbb{R}^n$.

Theorem 4.2. *The vectors*

$$\{q_{r+1}, \dots, q_n\}$$

form an orthonormal basis of $L^\perp$.

Proof

Because the full collection is orthonormal,

$$\langle q_j, q_i \rangle = 0$$

whenever

$$j > r, \quad i \leq r.$$

Every $y \in L$ can be written as

$$y = \sum_{i=1}^r a_i q_i.$$

Thus for $j > r$,

$$\langle q_j, y \rangle = \sum_{i=1}^r a_i \langle q_j, q_i \rangle = 0.$$

Hence

$$q_j \in L^\perp$$

for every $j = r + 1, \dots, n$.

It remains to show that these vectors span $L^\perp$.

Let $x \in L^\perp$. Since

$$\{q_1, \dots, q_n\}$$

is an orthonormal basis of $\mathbb{R}^n$,

$$x = \sum_{i=1}^n \langle q_i, x \rangle q_i.$$

But $x \in L^\perp$, so for every $i \leq r$,

$$\langle q_i, x \rangle = 0.$$

Therefore,

$$x = \sum_{i=r+1}^n \langle q_i, x \rangle q_i.$$

Thus

$$x \in \text{span}\{q_{r+1}, \dots, q_n\}.$$

Hence

$$\{q_{r+1}, \dots, q_n\}$$

is an orthonormal basis of $L^\perp$. $\square$

Therefore,

$$\boxed{\dim(L^\perp) = n - \dim(L)}.$$

Furthermore,

$$L \perp L^\perp$$

and

$$\dim(L) + \dim(L^\perp) = n.$$

Consequently,

$$\boxed{\mathbb{R}^n = L \oplus L^\perp}.$$

Thus every $x \in \mathbb{R}^n$ can be written uniquely as

$$x = y + z, \quad y \in L, \quad z \in L^\perp.$$

This decomposition will lead directly to the idea of an orthogonal projection.

—