

Lecture 10: Sums of Subspaces

Throughout this lecture, all vector spaces are over $\mathbb{R}$.

1 Sum of Two Subspaces

We begin with a natural way to combine two subspaces.

Definition. Let $U, V \subseteq \mathbb{R}^n$ be linear subspaces. Their sum is

$$U + V := \{u + v : u \in U, v \in V\}.$$

Thus, a vector x belongs to $U + V$ if and only if it can be written as

$$x = u + v$$

for some $u \in U$ and $v \in V$.

Since $0 \in U$ and $0 \in V$,

$$U \subseteq U + V, \quad V \subseteq U + V.$$

Indeed, if $u \in U$, then

$$u = u + 0 \in U + V,$$

and similarly for V .

Theorem 1.1. If $U, V \subseteq \mathbb{R}^n$ are linear subspaces, then $U + V$ is also a linear subspace of $\mathbb{R}^n$.

Proof:

First,

$$0 = 0 + 0 \in U + V.$$

Now let

$$x = u_1 + v_1, \quad y = u_2 + v_2$$

be two vectors in $U + V$, where $u_1, u_2 \in U$ and $v_1, v_2 \in V$.

Let $\alpha, \beta \in \mathbb{R}$. Then

$$\alpha x + \beta y = \alpha(u_1 + v_1) + \beta(u_2 + v_2)$$

and therefore

$$\alpha x + \beta y = (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2).$$

Since U and V are subspaces,

$$\alpha u_1 + \beta u_2 \in U$$

and

$$\alpha v_1 + \beta v_2 \in V.$$

Hence

$$\alpha x + \beta y \in U + V.$$

Therefore $U + V$ is a linear subspace of $\mathbb{R}^n$. $\square$

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2 Dimension of a Sum of Subspaces

Because

$$U \subseteq U + V \quad \text{and} \quad V \subseteq U + V,$$

we immediately have

$$\dim(U + V) \geq \max\{\dim(U), \dim(V)\}.$$

There is also the upper bound

$$\dim(U + V) \leq \dim(U) + \dim(V).$$

A more precise formula explains exactly how much overlap between U and V reduces the dimension.

Theorem: Dimension Formula

Let $U, V \subseteq \mathbb{R}^n$ be linear subspaces. Then

$$\boxed{\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)}.$$

Proof

Let

$$k = \dim(U \cap V).$$

Choose a basis

$$\{w_1, \dots, w_k\}$$

of $U \cap V$.

Since $U \cap V \subseteq U$, we can extend this basis to a basis of U :

$$\{w_1, \dots, w_k, u_{k+1}, \dots, u_r\},$$

where

$$r = \dim(U).$$

Similarly, extend the same basis of $U \cap V$ to a basis of V :

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_s\},$$

where

$$s = \dim(V).$$

We claim that

$$\mathcal{B} = \{w_1, \dots, w_k, u_{k+1}, \dots, u_r, v_{k+1}, \dots, v_s\}$$

is a basis of $U + V$.

Step 1: $\mathcal{B}$ spans $U + V$

Take $x \in U + V$. Then

$$x = u + v$$

for some $u \in U$ and $v \in V$.

Because the vectors above form bases of U and V ,

$$u = \sum_{i=1}^k a_i w_i + \sum_{i=k+1}^r b_i u_i$$

and

$$v = \sum_{i=1}^k c_i w_i + \sum_{i=k+1}^s d_i v_i.$$

Hence

$$x = \sum_{i=1}^k (a_i + c_i) w_i + \sum_{i=k+1}^r b_i u_i + \sum_{i=k+1}^s d_i v_i.$$

Thus $x \in \text{span}(\mathcal{B})$.

Therefore,

$$U + V \subseteq \text{span}(\mathcal{B}).$$

The reverse inclusion is immediate because every vector in $\mathcal{B}$ belongs to either U or V , and hence belongs to $U + V$.

Thus,

$$U + V = \text{span}(\mathcal{B}).$$

Step 2: $\mathcal{B}$ is linearly independent

Suppose

$$\sum_{i=1}^k a_i w_i + \sum_{i=k+1}^r b_i u_i + \sum_{i=k+1}^s c_i v_i = 0.$$

Rearranging,

$$\sum_{i=k+1}^s c_i v_i = - \left(\sum_{i=1}^k a_i w_i + \sum_{i=k+1}^r b_i u_i \right).$$

The right-hand side belongs to U , while the left-hand side belongs to V . Therefore,

$$\sum_{i=k+1}^s c_i v_i \in U \cap V.$$

Since $\{w_1, \dots, w_k\}$ is a basis of $U \cap V$, there exist scalars $d_1, \dots, d_k$ such that

$$\sum_{i=k+1}^s c_i v_i = \sum_{i=1}^k d_i w_i.$$

Therefore,

$$\sum_{i=1}^k (-d_i)w_i + \sum_{i=k+1}^s c_i v_i = 0.$$

But

$$\{w_1, \dots, w_k, v_{k+1}, \dots, v_s\}$$

is a basis of V , so it is linearly independent. Hence

$$c_{k+1} = \dots = c_s = 0.$$

Returning to the original equation gives

$$\sum_{i=1}^k a_i w_i + \sum_{i=k+1}^r b_i u_i = 0.$$

Since

$$\{w_1, \dots, w_k, u_{k+1}, \dots, u_r\}$$

is a basis of U , all the remaining coefficients are also zero.

Thus $\mathcal{B}$ is linearly independent.

Therefore,

$$\dim(U + V) = k + (r - k) + (s - k) = r + s - k.$$

That is,

$$\boxed{\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)}.$$

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Consequences

Since

$$\dim(U \cap V) \geq 0,$$

we obtain

$$\dim(U + V) \leq \dim(U) + \dim(V).$$

Furthermore,

$$\dim(U + V) = \dim(U) + \dim(V)$$

if and only if

$$U \cap V = \{0\}.$$

If, say,

$$\dim(U) \geq \dim(V),$$

then

$$\dim(U + V) = \dim(U)$$

if and only if

$$V \subseteq U.$$

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Direct Sums

Definition. *If*

$$U \cap V = \{0\},$$

*we call $U + V$ a **direct sum** and write*

$$U \oplus V.$$

An important feature of a direct sum is that the decomposition of a vector is unique.

Theorem 2.1. *If $U \cap V = \{0\}$, then every $x \in U + V$ can be written uniquely as*

$$x = u + v, \quad u \in U, \quad v \in V.$$

Proof

Suppose

$$x = u_1 + v_1 = u_2 + v_2.$$

Then

$$u_1 - u_2 = v_2 - v_1.$$

The left-hand side belongs to U , while the right-hand side belongs to V . Thus

$$u_1 - u_2 \in U \cap V.$$

Since $U \cap V = \{0\}$,

$$u_1 - u_2 = 0.$$

Therefore $u_1 = u_2$, and consequently $v_1 = v_2$. $\square$

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