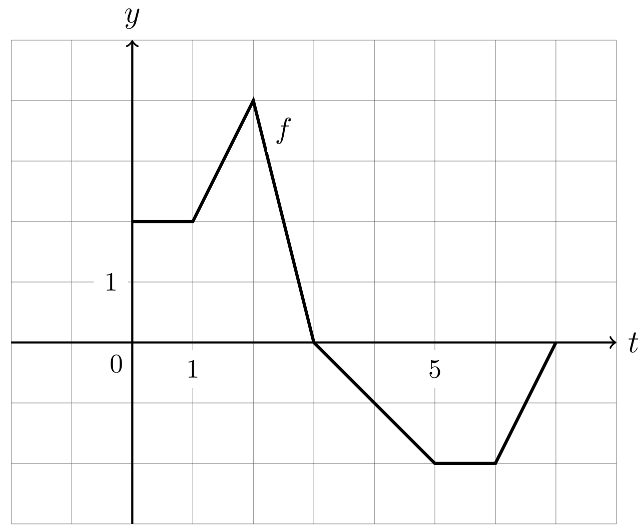
4.3 Problems

Question 1. Let $g(x) = \int_0^x f(t)\, dt$, where f is the function whose graph is shown.

(a) Evaluate $g(0), g(1), g(2), g(3)$, and $g(6)$.



$g(0) = 0$

$g(1) = 2$

$g(2) = 2 + 2 + 1 = 5$

$g(3) = 5 + \frac{1}{2} \cdot 4 \cdot 1 = 7$

$g(6) = 7 - \frac{1}{2} \cdot 2 \cdot 2 - 2 \cdot 1$

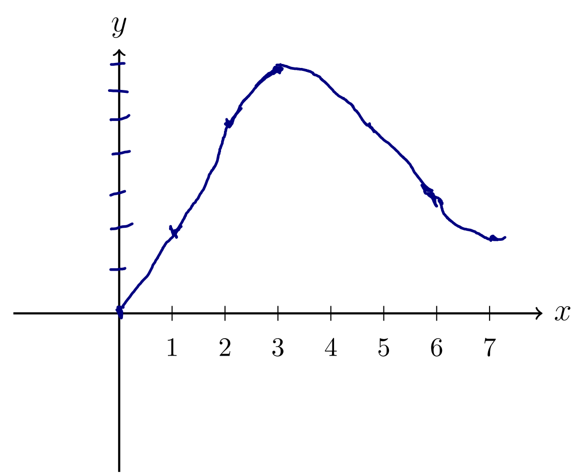
$= 3$

(b) On what interval is g increasing?

$[0, 3]$

(c) Where does g have a maximum value?

at $x = 3$



(d) Sketch a rough graph of g.

use part (a)

Question 2. Evaluate the integral:

(a) $\int_1^2 (8x^3 + 3x^2)\, dx$ $= F(2) - F(1) = (2\cdot 2^4 + 2^3) - (2\cdot 1^4 + 1^3) = 37$

F is any antiderivative of the integrand $8x^3 + 3x^2$

so $F = 8\cdot\frac{1}{4}x^4 + 3\cdot\frac{1}{3}x^3 = 2x^4 + x^3$

(b) $\int_0^9 \frac{\sqrt{u} - 2u^2}{u}\, du$ $= F(9) - F(0) = (2\cdot 3 - 81) - 0 = -75$

F is any antiderivative of the integrand $\frac{\sqrt{u} - 2u^2}{u} = u^{-\frac{1}{2}} - 2u$

so $F(u) = \frac{1}{\frac{1}{2}} u^{\frac{1}{2}} - u^2$

(c) $\int_0^3 |t^2 - 4|\, dt$

$$|t^2 - 4| = \begin{cases} t^2 - 4, & \text{if } t \geq 2 \text{ or } t \leq -2 \\ -t^2 + 4, & \text{if } t \in (-2, 2) \end{cases}$$

break the integration intervals into two parts

$$\int_0^3 |t^2 - 4|\, dt = \int_0^2 |t^2 - 4|\, dt + \int_2^3 |t^2 - 4|\, dt$$
$$= \int_0^2 (t^2 - 4)\, dt + \int_2^3 (-t^2 + 4)\, dt$$
$$= F(2) - F(0) + G(3) - G(2)$$
$$= 23/3$$

F is the antiderivative of $-t^2 + 4$

G is the antiderivative of $t^2 - 4$

so $F = -\frac{1}{3}t^3 + 4t$, $G = \frac{1}{3}t^3 - 4t$

(d) $\int_0^1 (1 + 2x)^3\, dx$

$F'(x) = (1 + 2x)^3$

$F(x) = \frac{1}{8}(1 + 2x)^4$

$$\int_0^1 (1 + 2x)^3\, dx = F(1) - F(0) = \frac{1}{8}\cdot 3^4 - \frac{1}{8}$$
$$= 10$$

Question 3. Find the derivative of the function:

(a) $F(x) = \int_0^{\pi} \sin\frac{t}{2}\cos\frac{t}{3}\,dt$

$F'(x) = 0$ as $F(x)$ is not changing with x

(b) $G(x) = \int_0^{x} \sin\frac{t}{2}\cos\frac{t}{3}\,dt$

$G'(x) = \sin\frac{x}{2}\cos\frac{x}{2}\,dt$

(c) $H(x) = \int_1^{\sin x} \frac{1-t^2}{1+t^4}\,dt$ let $f(x) = \int_1^x \frac{1-t^2}{1+t^2}\,dt$ then $f'(x) = \frac{1-x^2}{1+x^2}$

$= f(\sin x)$

$H'(x) = f'(\sin x)\cos x$

$= \frac{1-\sin^2 x}{1+\sin^2 x}\cdot\cos x$

(d) $J(x) = \int_{2x}^{3x+1} \sin(t^4)\,dt$ $= \int_0^{3x+1}\sin(t^4)dt - \int_0^{2x}\sin(t^4)\,dt$

let $f(x) = \int_0^x \sin(t^4)dt$

then $f'(x) = \sin(x^4)$

$= f(3x+1) - f(2x)$

so $J'(x) = f'(3x+1)\cdot 3 - f'(2x)\cdot 2$

$= \sin(3x+1)^4\cdot 3 - \sin(2x)^4\cdot 2$

Question 4. If $f(x) = \int_0^x (1-t^2)\cos^2 t\, dt$, on what interval is f increasing?

$$(1-t^2)\cos^2 t \geq 0$$

$$\Rightarrow t^2 \leq 1$$

$$\Rightarrow t \in (-1,1)$$

Question 5. On what interval is the curve $y = \int_0^x \frac{t^2}{t^2+t+2}\, dt$ concave downward?

$$y'(x) = \frac{x^2}{x^2+x+2}$$

$$y''(x) = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2}$$

concave downward $\Rightarrow y'' < 0 \Rightarrow x^2+4x < 0$

$$x(x+4) < 0$$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -4 & & 0 & \end{array} \quad x(x+4)$$

$$\Rightarrow -4 < x < 0$$