4.1 Problems

Table and Graph Problems

Example 1. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1	1.5	2	2.5	3
v (ft/s)	0	3	4	5	5.5	6	8

$\Delta t = 0.5$

upper sum $= 0.5(3+4+5+5.5+6+8) = 63/4$

lower sum $= 0.5(0+3+4+5+5.5+6) = 47/4$

Example 2. Speedometer readings for a motorcycle at 12-second intervals are given in the table.

t (s)	0	12	24	36	48	60
v (ft/s)	5	10	20	22	30	25

(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

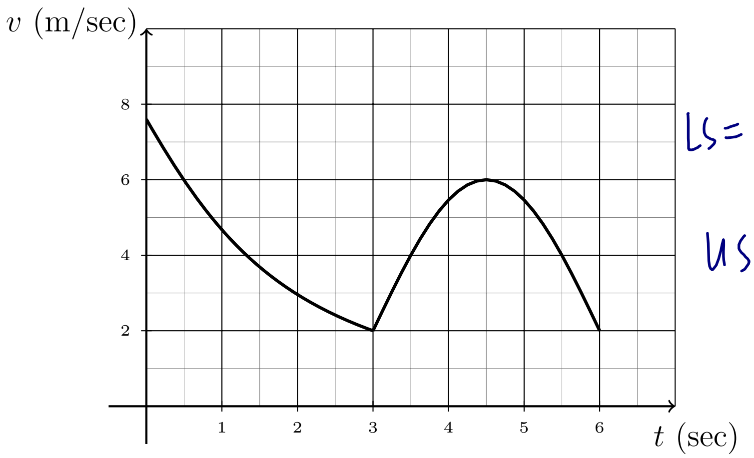
(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain. No

left sum $= 12(5+10+20+22+30) = 12 \cdot 87 = 1044$

right sum $= 12(10+20+22+30+25) = 12 \cdot 97 = 1164$

Example 3. The velocity graph of a particle is shown. Use it to give an overestimate and underestimate of the distance traveled by the particle on the interval $[0, 6]$.

upper sum lower sum



$LS = 1\,(4.6 + 3 + 2 + 2 + 5.5 + 2) = 19.1$

$US = 1\,(7.5 + 4.6 + 3 + 5.5 + 6 + 5.5)$

$= 32.1$

Examples with equations

Example 4. Estimate the area under the curve $f(x) = x/(x + 2)$ on $[1, 4]$ using 3 rectangles and

$=n$

(a) Left sums

(b) Right sums

$\Delta x = \frac{4-1}{3} = 1$

$x_0 = a = 1$

$x_1 = x_0 + \Delta x = 1 + 1 = 2$

$x_2 = x_1 + \Delta x = 2 + 1 = 3$

$x_3 = 3 + 1 = 4$

subintervals: $[1, 2], [2, 3], [3, 4]$

$LS = 1\,(f(1) + f(2) + f(3))$

$= 1 \cdot \left(\frac{1}{3} + \frac{1}{2} + \frac{3}{5}\right) \approx 1.43$

$RS = 1\,(f(2) + f(3) + f(4))$

$= 1 \cdot \left(\frac{1}{2} + \frac{3}{5} + \frac{2}{3}\right) \approx 1.77$

Example 5. Estimate the area under the curve $f(x) = \dfrac{1-x}{\sqrt{-x}}$ on $[-4,-1]$ using 4 rectangles and right sums.

$$\Delta x = \frac{-1-(-4)}{4} = \frac{3}{4}$$

$x_0 = a = -4$

$x_1 = -4 + \frac{3}{4} = -3.25$

$x_2 = -4 + 2\cdot\frac{3}{4} = -2.5$

$x_3 = -4 + 3\cdot\frac{3}{4} = -1.75$

$x_4 = -4 + 4\cdot\frac{3}{4} = -1$

$$RS = \frac{3}{4}\left(f(-3.25) + f(-2.5) + f(-1.75) + f(-1)\right)$$

≈ 6.65

Unseen Difficulties

Example 6. Consider the function $f(x) = x(x-4)^2$ on the interval $[0,6]$.

(a) Estimate the area under f using left-hand end points and

- (i) $n = 3$
- (ii) $\Delta x = 1$

(i) $n=3 \Rightarrow \Delta x = \frac{6}{3} = 2$

$LS = 2\left(f(0) + f(2) + f(4)\right) = 16$

(ii) $\Delta x = 1 \Rightarrow x_0 = 0,\ x_1 = 1\ \ x_2 = 2\ \ x_3 = 3,\ x_4 = 4\ \ x_5 = 5,\ x_6 = 6.$

$LS = 1\cdot\left(f(0) + f(1) + f(2) + f(3) + f(4) + f(5)\right)$

$= 1\cdot(0 + 9 + 8 + 3 + 0 + 5)$

$= 25$

(b) Estimate the area under f using right-hand end points and

(i) $\Delta x = 2$

(ii) $n = 6$

i) $\Delta x = 2 \Rightarrow [0,2]\,[2,4]\,[4,6]$

$RS = \Delta x(f(2)+f(4)+f(6)) = 2\cdot(8+0+24) = 64$

ii) $n=6 \Rightarrow \Delta x = \frac{6-0}{6} = 1$, $[0,1]\,[1,2]\,[2,3]\,[3,4]\,[4,5]\,[5,6]$

$RS = \Delta x(f(1)+f(2)+f(3)+f(4)+f(5)+f(6))$

$= 49$

(c) Use your curve sketching abilities and 4.1 video notes to explain why none of the previous estimates are technically upper sums.

See (d)

$f = x(x-4)^2$ on $[0,6]$

$f'(x) = (x-4)^2 + 2x(x-4)$

$(x-4)(x-4+2x) = 0$

$(x-4)(3x-4) = 0$

$x = \underline{4}, \underline{4/3}$

(d) Find an upper sum of f using 3 rectangles.

$n=3 \quad \Delta x = \frac{6}{3} = 2 \quad [0,2]\,[2,4]\,[4,6]$

$US = \Delta x\left(\max_{x\in[0,2]} f(x) + \max_{x\in[2,4]} f(x) + \max_{x\in[4,6]} f(x)\right) = \boxed{2\left(\frac{256}{27}+8+24\right)}$

$\max_{x\in[0,2]} f(x) = 256/27$

$f(\tfrac{4}{3}) = 256/27$

$f(0) = 0$

$f(2) = 8$

$\max_{x\in[2,4]} f(x) = 8$

$f(2) = 8$

$f(4) = 0$

$\max_{x\in[4,6]} f(x) = 24$

$f(4) = 0$

$f(6) = 24$