3.4 Problems

Level 1 Problems

Question 1. Evaluate the limit if it exists:

(a) $\lim_{x\to\infty} \frac{x+1}{3-2x}$ $= \lim_{x\to\infty} \frac{x+1}{3-2x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x} \to 0}{\underbrace{\frac{3}{x}}_{0} - 2} = -\frac{1}{2}$$

(b) $\lim_{x\to\infty} \frac{x^2+x-5}{3x-2\sqrt{x^3}}$ $= \lim_{x\to\infty} \frac{x^2+x-5}{3x-2\sqrt{x^3}} \cdot \frac{\frac{1}{x^{3/2}}}{\frac{1}{x^{3/2}}}$ ← highest power in the denominator

$$= \lim_{x\to\infty} \frac{\sqrt{x} + \frac{1}{\sqrt{x}} \to 0 - \frac{5}{x^{3/2}} \to 0}{\underbrace{\frac{3}{\sqrt{x}}}_{0} - 2} = \lim_{x\to\infty} \frac{\sqrt{x}}{-2} = -\infty$$

(c) $\lim_{x\to-\infty} \frac{x^2+x-5}{3x-2\sqrt{x^3}}$ ← not defined for negative x

Quick Conceptual/Fun Questions

Question 2. What is the maximum number of vertical asymptotes that a function can have?

as many as you want

Question 3. What is the maximum number of horizontal asymptotes that a function can have?

two

domain $(x>0)$

Question 4. Give an example of a function $g(x)$ so that $f(x) = \dfrac{3\sqrt[3]{x^2} + 5\sqrt{x^3} + 7\sqrt[5]{x^9}}{xg(x)}$ has a horizontal asymptote $y = 6$. $\Rightarrow$

$$\lim_{x\to\infty} f(x) = 6$$

highest power

$$\Leftrightarrow \lim_{x\to\infty} \frac{3\sqrt[3]{x^2}+5\sqrt{x^3}+7\sqrt[5]{x^9}}{x\,g(x)} = 6$$

$$\Rightarrow \lim_{x\to\infty} \frac{0+0+7x^{9/5}}{x\ g(x)} = 6$$

$$\Leftrightarrow \lim_{x\to\infty} \frac{7x^{4/5}}{g(x)} = 6$$

If $g(x)$ has an order $> 4/5$ then the limit on the left hand side is $0 \neq 6$; if $g(x)$ has an order $< 4/5$, then the limit is $\pm\infty$. For the limit to $=6$, $g(x) = c\,x^{4/5}$

$$\lim_{x\to\infty} \frac{7x^{4/5}}{cx^{4/5}} = 6 \Rightarrow c = 7/6$$

Level 2+ Problems

Question 5. Find the horizontal asymptote(s) of the following functions if they exist.

(a) $f(x) = \dfrac{(x+1)(2x-5)}{(3x-1)(1-x)}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+1)(2x-5)}{(3x-1)(1-x)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$ highest order in the denominator

$$= \lim_{x\to\infty} \frac{(1+\frac{1}{x})(2-\frac{5}{x})}{(3-\frac{1}{x})(\frac{1}{x}-1)} = \lim_{x\to\infty} \frac{1\cdot 2}{3\cdot(-1)} = -\frac{2}{3}$$

similarly $\lim_{x\to-\infty} f(x) = -2/3$. So H.A. is $y = -2/3$

(b) $g(x) = \dfrac{(x+1)(2x-5)}{|3x^2|}$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{(x+1)(2x-5)}{3x^2} = \lim_{x\to\infty} \frac{1}{3}(1+\underbrace{\frac{1}{x}}_{\to 0})(2-\underbrace{\frac{5}{x}}_{\to 0}) = \frac{2}{3}$$

similarly $\lim_{x\to-\infty} g(x) = 2/3$

so H.A. is $y = 2/3$

(c) $h(x) = \dfrac{\sqrt{3x^6+5x+1}}{(x^2+1)(2x-5)}$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\sqrt{3x^6+5x+1}}{(x^2+1)(2x-5)} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

as $x>0$

$$= \lim_{x\to\infty} \frac{\sqrt{3x^6+5x+1}}{(x^2+1)(2x-5)} \cdot \frac{\frac{1}{\sqrt{x^6}}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3+\frac{5}{x^5}+\frac{1}{x^6}}}{(1+\frac{1}{x^2})(2-\frac{5}{x})} = \frac{\sqrt{3}}{2}$$

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{\sqrt{3x^6+5x+1}}{(x^2+1)(2x-5)} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

as $x<0$

$$= \lim_{x\to-\infty} \frac{\sqrt{3x^6+5x+1}}{(x^2+1)(2x-5)} \cdot \frac{\left(-\frac{1}{\sqrt{x^6}}\right)}{\frac{1}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{3+\frac{5}{x^5}+\frac{1}{x^6}}}{(1+\frac{1}{x^2})(2-\frac{5}{x})}$$

$$= -\sqrt{3}/2$$

so H.A. are $y = \sqrt{3}/2$, $y = -\sqrt{3}/2$

Question 6. Find the horizontal asymptote(s) of the following functions if they exist.

(a) $f(x) = \sqrt{9x^2 + x} - 3x$

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{\cancel{9x^2}+x-\cancel{9x^2}}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3} = \boxed{\frac{1}{6}}$$

H.A. is $y = 1/6$

$$\lim_{x\to-\infty}\left(\sqrt{9x^2+x}-3x\right) = \lim_{x\to-\infty}\sqrt{9x^2}-3x = \lim_{x\to-\infty} -3x-3x = \infty$$

(b) $g(x) = \sqrt{x^2 + x + 1} + x$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(\sqrt{x^2+x+1}+x\right) \underset{\text{x is the highest power}}{=\!=\!=} \lim_{x\to\infty}\sqrt{x^2+0+0}+x = \lim_{x\to\infty}2x = \infty$$

$$\lim_{x\to-\infty}\sqrt{x^2+x+1}+x = \lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\cdot\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x} = \lim_{x\to-\infty}\frac{\cancel{x^2}+x+1-\cancel{x^2}}{\sqrt{x^2+x+1}-x}$$

$\sqrt{x^2} = -x$

so x is not the highest power

$$= \lim_{x\to-\infty}\frac{x+1}{\sqrt{x^2+x+1}-x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{\sqrt{x^2+x+1}}{x}-1} = \lim_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{\sqrt{x^2+x+1}}{-\sqrt{x^2}}-1}$$

as $x<0$

$$= \lim_{x\to-\infty}\frac{1+\cancel{\frac{1}{x}}}{-\sqrt{1+\cancel{\frac{1}{x}}+\cancel{\frac{1}{x^2}}}-1} = -\frac{1}{2}$$

(c) $h(x) = 3x \sin \dfrac{2}{5x}$

$$\lim_{x\to\infty} 3x\sin\boxed{\frac{2}{5x}}^{=v} \quad (1)$$

change of variable

$v = \dfrac{2}{5x}$,

which means $x = \dfrac{2}{5v}$

as $x\to\infty$, $v\to 0$

use this to replace every x in (1) by v

$$= \lim_{v\to 0} 3\cdot\frac{2}{5v}\sin v \quad (\to 1)$$

$$= \frac{6}{5}$$

the change of variable helps to simplify the expression inside the sine function!!