1.8 Problems

Example 1. Determine where the following functions are continuous.

(a) $f(x) = \begin{cases} 3 & \text{if } x = -1 \\ 5 & \text{if } x = -1/2 \\ \dfrac{x^2 - x - 2}{2x^2 + 3x + 1} & \text{otherwise} \end{cases}$

Solution: Let us simplify the third expression first $\dfrac{x^2 - x - 2}{2x^2 + 3x + 1} = \dfrac{(x-2)(x+1)}{(x+1)(2x+1)} = \dfrac{x-2}{2x+1}$. From this we know that $x = -1/2$ is a V.A., so there is an infinite discontinuity at $\boxed{x = -1/2}$. By the definition of this function, there is possibly a jump discontinuity at $x = -1$, so we need to check it. Observe that $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \dfrac{x-2}{2x+1} = 3$, and in the above definition $f(-1) = 3$. These imply $\lim\limits_{x \to -1} f(x) = f(-1)$, f is continuous at -1. $f(x)$ is continuous on $\boxed{(-\infty, -1/2) \cup (1/2, \infty)}$.

(b) $g(x) = \dfrac{2 \tan x}{\sin x + \cos x}$

Solution: rewrite

$$g(x) = \frac{2 \tan x}{\sin x + \cos x} = \frac{2 \sin x}{\cos x(\sin x + \cos x)}$$

$g(x)$ has a denominator, so it possibly has V.A. . The denominator equals 0 if and only if one or both of its factors equal to 0. The factor $\cos x = 0$ if and only if $\boxed{x = \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi}$, where k is any integer, and the factor $\sin x + \cos x = 0$ if and only if $\tan x = -1$, which means $\boxed{x = \pi/4 + k\pi}$ where k is any integer. $f(x)$ is continuous at all other points.

(c) $h(x) = \sqrt{\dfrac{x^2 - x - 6}{x}}$

Solution: $x = 0$ is a V.A., so there is an infinite discontinuity at $x = 0$. In order for the function to be well defined, we need (1) $x \neq 0$ (as $x = 0$ makes the denominator 0), (2) $\dfrac{x^2 - x - 6}{x} \geq 0$, things under the square root needs to be nonnegative. Solving the inequality $\dfrac{x^2 - x - 6}{x} \geq 0$, we obtain $x \in [-2, 0] \cup [3, \infty)$. As $x \neq 0$ due to (1), the domain of $h(x)$ is $[-2, 0) \cup [3, \infty)$. Based on the expression of $h(x)$, there is no jump or removable discontinuity, so the function is continuous on $\boxed{[-2, 0) \cup [3, \infty)}$.

Example 2. For what value of the constant c is the function

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

continuous on $(-\infty, \infty)$.

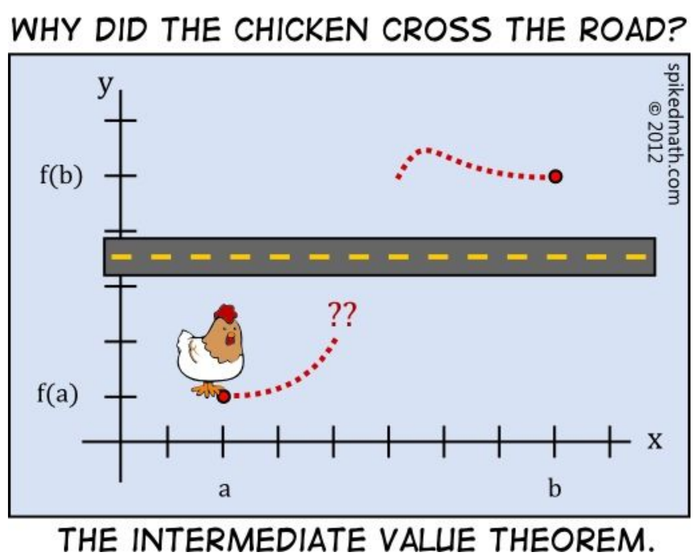
Solution: based on the expression of $f(x)$, only jump discontinuity can exist and if exists, it must be at $x = 2$. Therefore, if we could make sure $f(x)$ is continuous at $x = 2$, then $f(x)$ would be continuous on $(-\infty, \infty)$. We would make $f(x)$ continuous at $x = 2$ by setting c to a property value. For $f(x)$ to be continuous at $x = 2$, we need

$$\begin{aligned} & \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) \\ \iff & \lim_{x\to 2^-} (cx^2 + 2x) = \lim_{x\to 2^+} (x^3 - cx) = 2^3 - 2c \\ \iff & 4c + 4 = 8 - 2c \\ \iff & c = \boxed{2/3}. \end{aligned}$$

Example 3. State the Intermediate Value Theorem (from heart.)

Example 4. Using the Intermediate Value Theorem explain why the chicken crossed the road (in the below picture)

Solution: since the chicken cannot teleport, its route must be a continuous curve. Since the curve started from one side of the road and ended on the other side, IVT says it must cross the road at some point.



Example 5. Use the Intermediate Value Theorem to show that there is a root of the given equation

$$\sqrt[3]{x} + x - 1 = 0.$$

Solution: Let $f(x) = \sqrt[3]{x} + x - 1$. This problem asks us to show $f(x) = 0$ has a solution. In other words, the curve of $f(x)$ must cross the x-axis. For this purpose, we need to find some a and b such that $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then IVT can ensure that starting from $(a, f(a))$ and ending at $(b, f(b))$, the graph of f must cross 0 at some point in between.
Try $f(1)$: $f(1) = 1 + 1 - 1 > 0$.
Try $f(-1)$: $f(-1) = -1 - 1 - 1 < 0$,
Since $f(x)$ is continuous everywhere on the $(-1, 1)$ and $f(-1) < 0 < f(1)$, by IVT, there is $c \in (-1, 1)$ such that $f(c) = 0$, or equivalently, $\sqrt[3]{c} + c - 1 = 0$.

Example 6. Use the Intermediate Value Theorem to show that there is a root of the given equation

$$\sqrt[3]{x} + x = 7.$$

Solution: Let $f(x) = \sqrt[3]{x} + x$. This problem asks us to show $f(x) = 7$ has a solution. In other words, the curve of $f(x)$ must cross $y = 7$. For this purpose, we need to find some a and b such that $f(a) < 7 < f(b)$ or $f(b) < 7 < f(a)$, then IVT can ensure that starting from $(a, f(a))$ and ending at $(b, f(b))$, the graph of f must cross $y = 7$ at some point in between. Try $f(1)$: $f(1) = 1 + 1 < 7$.
Try $f(27)$: $f(27) = 3 + 27 > 7$. Since $f(x)$ is continuous everywhere on the $(1, 27)$ and $f(1) < 7 < f(27)$, by IVT, there is $c \in (1, 27)$ such that $f(c) = 7$, or equivalently, $\sqrt[3]{x} + x = 7$.

Example 7. Use the Intermediate Value Theorem to show that there is a root of the given equation

$$\frac{1}{x+3} = \sqrt{x-5}.$$

Solution: Let us rewrite the above equation to something equals 0. For example, we can rewrite it as $\dfrac{1}{x+3} - \sqrt{x-5} = 0$. Then we proceeds as above. Let $f(x) = \dfrac{1}{x+3} - \sqrt{x-5}$. Then we need to show $f(x) = 0$ has a solution. In order to show $f(x)$ cross $y = 0$ somewhere, we try $f(5) = \frac{1}{8} > 0$ and $f(9) = \frac{1}{12} - 2 < 0$. In addition, $f(x)$ is continuous on $(5, 9)$, the IVT implies that there is a c s.t. (s.t. means such that) $f(c) = 0$.

Example 8. Suppose that f to be continuous everywhere with $f(1) = 5$, $f(3) = 2$, and $f(11) = -1$. Which of the following is necessarily a true statement?

A. $f(c) = 0$ for some $c \in [1, 3]$.

B. $f(c) = 0$ for some $c \in [-1, 5]$.

C. $f(a) = f(b)$ for some $a \neq b$.

D. $\boxed{f(c) = 4 \text{ for some } c \in [1, 3].}$

E. None of the above are true.