Wavelets and the angle between past and future

Sergei Treil and Alexander Volberg
Department of Mathematics
Michigan State University

Statement of the problem

$W = W^* \geq 0$ — $d \times d$ matrix weight on T

$L^2(W)$ — weighted L^2-space of C^d-valued functions

$$\|f\|^2_{L^2(W)} = \int_T (W(\xi)f(\xi), f(\xi))dm(\xi)$$

P_+ — Riesz Projection

$$P_+ f = \sum_{k=0}^{\infty} \hat{f}(k)z^k, \qquad \text{where } f = \sum_{k=-\infty}^{\infty} \hat{f}(k)z^k\,.$$

Question: When $P_+$is bounded in $L^2(W)$
or equivalently, when

$$\int_T (WP_+f, P_+f)dm \leq C \int_T (Wf, f)dm$$

$\forall f \in L^2(W)$.

Why “past and future”?

Consider spectral representation of a stationary process (with discrete time).

$W\,dm$ its spectral measure,
$\mathcal{L}(z^n : n < 0)$ its “past”,
$\mathcal{L}(z^n : n \geq 0)$ its future

The angle is small $\Longleftrightarrow$ the norm of projection is big,

$$\|\mathcal{P}\| = 1/\sin\alpha$$

Another motivation: Toeplitz operators

Consider a Toeplitz operator T_F with symbol F, T_F : $H^2(C^d) \to H^2(C^d)$,

$$T_F f = P_+ F f\,, \qquad f \in H^2(C^d)\,.$$

Here $F \in L^\infty(M_{d\times d})$

Well known that T_F is invertible iff $F = G_1^* G_2$, where $G_1,\ G_2 \in H^2(M_{d\times d})$, $G_1^{-1}, G_2^{-1} \in H^2$ and the operator

$$G_2^{-1} P_+ (G_1^{-1})^* \tag{1}$$

is bounded on H^2 or on L^2.

Equivalent to the two weight inequality

$$\int_T (VP_+ f, P_+ f) dm \le C \int_T (Wf, f) dm$$

where $W = G_1 G_1^*$, $V = (G_2^{-1})^* G_2^{-1}$.
But $G^{-1} \in L^\infty$, so $V \asymp W$, i.e.

$$\frac{1}{C} V \le W \le CV\,.$$

Scalar Case ($d = 1$)

Well known
Helson–Szegö Theorem. *P_+ is bounded in $L^2(w)$ iff $w = \exp(u + \tilde{v})$, $u, v \in L^\infty$, $\|v\|_\infty < \pi/2$*

Hunt–Muckenhoupt–Wheeden Theorem. *P_+ is bounded in $L^2(w)$ iff*

$$\sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-1} \right) < \infty \qquad (\mathsf{A}_2)$$

Main result

Theorem. *P_+ is bounded in $L^2(W)$ iff*

$$\sup_I \left\| \left(\frac{1}{|I|} \int_I W \right)^{1/2} \left(\frac{1}{|I|} \int_I W^{-1} \right)^{1/2} \right\| < \infty \qquad (\mathsf{v}A_2)$$

Necessity

Consider projections

$$P_\lambda f = P_+ f - b_\lambda P_+ \overline{b}_\lambda f = k_\lambda(z) \int_T f(\xi)\overline{k_\lambda}(\xi)dm(\xi)$$

where b_λ is a Blaschke factor, k_λ is a normalized reproducing kernel of H^2

$$k_\lambda(z) = \frac{(1-|z|^2)^{1/2}}{1-\overline{\lambda}z}$$

Note that $|k_\lambda|^2$ is the Poisson kernel.

Easy to compute

$$\|P_\lambda\| = \left\| \Big(\int_T W|k_\lambda|^2 \Big)^{1/2} \Big(\int_T W^{-1}|k_\lambda|^2 \Big)^{1/2} \right\| =$$
$$= \left\| \Big(W(\lambda) \Big)^{1/2} \left(W^{-1}(\lambda) \right)^{1/2} \right\| \quad (2)$$

And

$$C \cdot W(\lambda_I) \geq W_I = \frac{1}{|I|} \int_I W dm$$

where C is an absolute constant.
So (**??**) implies (vA_2).

Sufficiency.

Why the Calderon–Zygmund theory does not work?

There is no maximal function.

Example: Scalar case: If $0 \le K_1(s,t) \le K_2(s,t)$, and integral operator with kernel K_2 is bounded, then the operator with kernel K_1 is also bounded in scalalr $L^2(w)$.

Not true for $L^2(W)$ (even for scalar K_1, K_2).

Main step: Haar system is an unconditional basis (Riesz basis) in $L^2(W)$. That means

$$f = \sum_{\text{dyadic } I} h_I e_I, \quad e_I \in C^d$$

and

$$\|f\|_{L^2(W)} \asymp \Big(\sum_{\text{dyadic } I} \|h_I e_I\|^2_{L^2(W)} \Big)^{1/2}$$

Here h_I denotes the Haar function on I

So

$$\|h_I e\|^2_{L^2(W)} = \int_I (W(\xi)e, e) dm(\xi) =$$
$$= |I| \cdot (W_I e, e)$$

$\asymp \iff$ two inequalities $\iff$ two "Imbedding Theorems".

Since W and W^{-1} are both Muckenhoupt, we need prove only one "Imbedding Theorem".

Why imbedding theorem? (scalar case)

Well known: A *uniformly minimal* and almost normalized system of vectors f_n is a Riesz basis iff

$$\sum_n |(f, f_n)|^2 \leq C\|f\|^2$$

and the same holds for dual (biorthogonal system f_n'.
Dual to Haar system in $L^2(w)$ — Haar system in $L^2(1/w)$

Let h_I^w be normalized in $L^2(w)$ Haar function

$$h_I^w = w(I)^{-1} h_I \quad \text{where } w(I) = \int_I w$$

Pick μ_I s.t. $\tilde{h}_I^w = h_I^w + \mu_I \chi_I$ is orthogonal system.

System $\tilde{h}_I^w$ is almost normalized, so enough to show

$$\sum_I |\mu_I|^2 |(fw)(I)|^2 =$$

$$= \sum_I |\mu_I|^2 |I|^2 |(fw)_I|^2 \le C \|f\|_w^2$$

or

$$\sum_I |\mu_I|^2 |I|^2 |f_I|^2 \le C \|f\|_{w^{-1}}^2$$

Almost normalized: $0 < \inf_n \|f_n\| \leq \sup_n \|f_n\| < \infty$.
Dual (biortogonal) system: f'_n s.t. $(f_n, f'_k) = 1$ if $n = k$ and $= 0$ if $n \neq k$.

Main tools:

1. Senichkin–Vinogradov Test. *Let $\mathcal{Z}$ be a measure space, k be a locally summable nonnegative measurable function on $\mathcal{Z} \times \mathcal{Z}$. Let*

$$\int_{\mathcal{Z}} k(s,t)k(s,x)ds \leq C\left[k(x,t) + k(t,x)\right] ,$$

a.e. on $\mathcal{Z} \times \mathcal{Z}$. Then for any positive $g \in L^2(\mathcal{Z})$

$$\iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)g(s)g(t)dsdt \leq 2C\|g\|^2_{L^2(\mathcal{Z})} ,$$

and the integral operator with kernel $k(s,t)$ is bounded in $L^2(\mathcal{Z})$.

2. Scalarization Lemma. *For a matrix weight* $W = W^* \geq 0$

$$\Big\| \int_I W \Big\| \asymp \int_I \|W\| \asymp \int_I \operatorname{trace} W$$

So the Muckenhoupt condition can be rewritten as

$$\frac{1}{|I|}\int_I \Big\| \big(W_I\big)^{1/2} W^{-1}(\xi) \big(W_I\big)^{1/2} \Big\| dm(\xi) \leq C$$

Proof of the Senichkin–Vinogradov Test.
Let $f \in L^2$, $\|f\|_2 = 1$, $f \geq 0$ be a function such that

$$Q = \iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)f(s)f(t)dsdt < \infty\,.$$

Then

$$Q^2 \stackrel{\mathsf{def}}{=} \left(\int_{\mathcal{Z}} f(s)\left(\int_{\mathcal{Z}} k(s,t)f(t)dt\right)ds\right)^2 \leq$$

$$\|f\|^2_{L^2(\mathcal{Z})} \cdot \left\|\int_{\mathcal{Z}} k(.,t)f(t)dt\right\|^2 \leq$$

$$\int_{\mathcal{Z}} ds \left(\iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)k(s,x)f(t)f(x)dtdx\right) =$$

$$\iint_{\mathcal{Z}\times\mathcal{Z}} f(t)f(x)\left(\int_{\mathcal{Z}} k(s,t)k(s,x)ds\right) \leq$$

$$2C\int_{\mathcal{Z}} [k(t,x)+k(x,t)]f(t)f(x)dtdx =$$

$$2CQ\,,$$

So

$$Q^2 \leq 2CQ\,.$$

Therefore $Q \leq 2C$

Bases of wavelets.

On R

Let φ, ψ be functions on R,
I — dyadic interval with left endpoint x_I and length $|I| = 2^{-\nu}$ define

$$\varphi_I(x) = |I|^{-1/2}\varphi(2^\nu(x - x_I)),$$

$$\psi_I(x) = |I|^{-1/2}\psi(2^\nu(x - x_I)).$$

Assume the system φ_I is a complete *minimal* system in L^2, and ψ_I is its dual (biorthogonal). Also assume

1. φ, ψ are bounded, and moreover $|\varphi(x)| \leq C/|x|^2$, $|\psi(x)| \leq C/|x|^2$ at ∞;
2. φ, ψ, are absolutely continuous and φ' and ψ' are bounded and satisfy the estimates
   $|\varphi'(x)| \leq C/|x|^2$, $|\psi'(x)| \leq C/|x|^2$ at ∞;

Then, if W is a Muckenhoupt weight, the wavelet subspaces

$$\Phi_I \stackrel{\text{def}}{=} \left\{ \varphi_I e \ : \ e \in C^d \right\}$$

form a Riesz basis in $L^2(W)$.

Note: Condition 1 implies

3. φ, $\psi \in L^1(R)$, and, moreover
$\int\limits_R^{\infty} |\varphi|,\ \int\limits_{-\infty}^{-R} |\varphi|,\ \int\limits_R^{\infty} |\psi|,\ \int\limits_{-\infty}^{-R} |\psi| \leq C/|R|$

Condition 1 — 3 imply

4. $\int_R \varphi(t)dt = \int_R \psi(t)dt = 0.$

Continuous wavelet decompositions.

Littlewood–Paley theory

Notation: f defined on T, $f(\lambda)$ is Poisson extension at $\lambda \in D$.
$f^{-1}(z)$ and $f(z)^{-1}$ are different.

An equivalent norm in L^2. *Let*

$$f \in L^2_0 \stackrel{\text{def}}{=} \left\{ f \in L^2(T) \ : \ f(0) = 0 \right\}.$$

Then

$$\int_0^{2\pi} |f(e^{it})|^2 \frac{dt}{2\pi} = \frac{1}{\pi} \iint_D |\nabla f(z)|^2 \log \frac{1}{|z|} \, dxdy \asymp \iint_D |\nabla f(z)|^2 (1 - |z|^2) \, dxdy$$

Continuous wavelet decomposition

An equivalent norm in $L^2(w)$. *Let w be a Muckenhoupt (A_2) weight,*

$$f \in L_0^2(w) \stackrel{\text{def}}{=} \left\{ f \in L^2(w) \ : \ f(0) = 0 \right\}.$$

Then

$$\int_T |f|^2 w \asymp \iint_D |\nabla f(z)|^2 w(z)(1 - |z|^2)\, dxdy$$

Proof of the Hunt–Muckenhoupt–Wheeden theorem

Since $|\nabla f|^2 = |\partial f|^2 + |\overline{\partial} f|^2$ the mapping

$$f \mapsto \begin{pmatrix} \partial f(z) \\ \overline{\partial} f(z) \end{pmatrix}$$

gives a *new representation* of $L_0^2(w)$ in the disk. P_+ is simply an (orthogonal) coordinate projection in this new representation.

A Weighted Imbedding Theorem

Theorem. *Let w satisfy (A_2), and μ be a Carleson measure on D. Then $\forall f \in L^2(T, w)$,*

$$\iint_D |f(z)|^2 w(z)\, d\mu(z) \le C \int_T |f|^2 w\,;$$

C depends on Carleson norm of μ and Muckenhoupt constant of w.

Carleson measure. If w satisfies (A_2), then the measure

$$\frac{|\nabla w(z)|^2}{w(z)^2}(1 - |z|^2)dxdy$$

is Carleson.

Carleson measures.

A measure μ on D is called Carleson if one of the following equivalent conditions hold

1. For any $f \in H^2$

$$\int_D |f(z)|^2 d\mu \leq C\|f\|^2$$

2. For any $f \in L^2(T)$

$$\int_D |f(z)|^2 d\mu \leq C\|f\|^2$$

3. $\sup\limits_{I \subset T} \dfrac{\mu(Q_I)}{|I|} < \infty$

4. $\sup\limits_{s \in D} \displaystyle\int_D \frac{1-|s|^2}{|1-\overline{s}z|^2} d\mu(z) < \infty$

Equivalent norm: the proof

Notation: $\langle .,. \rangle$ — standard *linear* duality,

$$\langle a, b \rangle = \sum a_k b_k.$$

Green's formula:

$$X \stackrel{\mathsf{def}}{=} \int_T |f|^2 w =$$

$$\frac{1}{2\pi} \iint_D \Delta\left(|f(z)|^2 w(z)\right) \log\frac{1}{|z|}\, dxdy =$$

$$= \frac{1}{2\pi} \iint_D |\nabla f(z)|^2 w(z) \log\frac{1}{|z|}\, dxdy +$$

$$+ \frac{1}{2\pi} \iint_D f(z)\langle \nabla \overline{f}(z), \nabla w(z)\rangle \log\frac{1}{|z|}\, dxdy +$$

$$\frac{1}{2\pi} \iint_D \overline{f}(z)\langle \nabla f(z), \nabla w(z)\rangle \log\frac{1}{|z|}\, dxdy =$$

$$= \frac{1}{2\pi} \iint_D |\nabla f(z)|^2 w(z) \log\frac{1}{|z|}\, dxdy +$$

$$\frac{1}{\pi} \mathrm{Re} \iint_D \overline{f}(z)\langle \nabla f(z), \nabla w(z)\rangle \log\frac{1}{|z|}\, dxdy$$

We can replace $\log\frac{1}{|z|}$ by $(1 - |z|^2)$.

Let us estimate the last integral:

$$\left| \iint\limits_D \overline{f}(z)\langle \nabla f(z), \nabla w(z)\rangle \log \frac{1}{|z|}\, dxdy \right| \leq$$

$$C \iint\limits_D |f(z)|\,|\nabla f(z)|\,|\nabla w(z)|(1-|z|^2)\, dxdy \leq$$

$$\leq C \left(\iint\limits_D |\nabla f(z)|^2 w(z)(1-|z|^2)\, dxdy \right)^{1/2} \times$$

$$\left(\iint\limits_D |f(z)|^2 w(z) \frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\, dxdy \right)^{1/2}$$

$$\leq C \left(\iint\limits_D |\nabla f(z)|^2 w(z)(1-|z|^2)\, dxdy \right)^{1/2} \times$$

$$\left(\int\limits_T |f|^2 w \right)^{1/2}$$

If we denote

$$X = \int_T |f|^2 w,$$
$$Y = \iint_D |\nabla f(z)|^2 w(z)(1 - |z|^2)\, dxdy\,,$$

then the above estimates give us

$$X \leq C(Y + (XY)^{1/2})$$

and

$$Y \leq C(X + (XY)^{1/2})\,.$$

Therefore X and Y are equivalent.

Let us prove that the measure

$$\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)dxdy \text{ **is Carleson:**}$$

The condition (A_2) can be rewritten as

$$\sup_{z\in D}(w(z))(w^{-1}(z)) < \infty\,.$$

Rewrite this as

$$\sup_{z\in D}\Big\{\Big[w(z)/\exp\left\{(\log w)(z)\right\}\Big]\times$$
$$\times\ \big[(w^{-1})(z)/\exp\big\{(\log w^{-1})(z)\big\}\big]\Big\} < \infty$$

By Jensen inequality expressions in brackets are at least 1, so

$$\sup_{z\in D}\left[w(z)/\exp\left\{(\log w)(z)\right\}\right] = C < \infty\,,$$

or equivalently

$$\log\left\{w(z)\right\} - \left(\log w\right)(z) \le C$$

By Green's formula

$$\log\{w(s)\} - (\log w)\,(s) =$$

$$-\frac{1}{2\pi}\iint_D \log\left|\frac{1-\overline{s}z}{z-s}\right| \Delta \log\{w(z)\}\ dxdy =$$

$$\frac{1}{2\pi}\iint_D \frac{|\nabla w(z)|^2}{w(z)^2} \log\left|\frac{1-\overline{s}z}{z-s}\right|\ dxdy =$$

$$= \frac{1}{4\pi}\iint_D \frac{|\nabla w(z)|^2}{w(z)^2} \log\left|\frac{1-\overline{s}z}{z-s}\right|^2\ dxdy$$

Using $\log(1/a) \geq 1-a$ for $0 < a \leq 1$ we estimate the last integral as

$$\iint_D \frac{|\nabla w(z)|^2}{w(z)^2} \log\left|\frac{1-\overline{s}z}{z-s}\right|^2 dxdy \geq$$

$$\geq \iint_D \frac{|\nabla w(z)|^2}{w(z)^2}\left(1-\left|\frac{1-\overline{s}z}{z-s}\right|^2\right) dxdy =$$

$$= \iint_D \frac{|\nabla w(z)|^2}{w(z)^2} \cdot \frac{(1-|s|^2)(1-|z|^2)}{|1-\overline{s}z|^2}\, dxdy$$

Together with $\log\{w(z)\} - (\log w)(z) \leq C$ this imply

$$\iint_D \frac{(1-|s|^2)}{|1-\overline{s}z|^2} \cdot \frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy \leq C\,,$$

$\forall s \in D$. That is simply a conformal invariant characterization of Carleson measures.

So the measure

$$\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy$$

is Carleson.

Remark: The measure $\dfrac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy$ is Carleson iff w satisfies the invariant (A_∞) condition discussed in the talk of Dechao Zheng.

Idea of the proof of the imbedding theorem.

Consider an imbedding operator
$\mathcal{J} : L^2(T) \to L^2(\mu)$,

$$(\mathcal{J}f)(z) = w(z)^{1/2} \left(w^{-1/2} f\right)(z).$$

To prove the theorem it is enough to show that the operator $\mathcal{J}$ is bounded.

Apply Senichkin–Vinogradov Test to $\mathcal{J}\mathcal{J}^*$

This idea of the proof is due to S. Vinogradov, St. Petersburg University. It was used by him to prove the classical Carleson imbedding theorem. For the Vinogradov's proof see Nikolskii, *Treatise on the shift operator*, Springer 1985.

Papers and transparencies of the talk available on my home page
http://www.mth.msu.edu/~treil
Algorithm:

1. start Netscape
2. in "file" menu chose "open location"
3. in the dialog box that appear type
   "**http://www.mth.msu.edu/~treil**"
   (without " ... ")
4. if you want to get transparencies of the talk click on "talks at MSRI"
5. if you want to get papers click on a title you want and go to the next step
6. to download hold "shift" key ("command" key on Mac) and click on an item you need.
7. If your browser support applications for viewing dvi or postscript files you can view them without downloading (simply click on the item). If it does not support them, you do not have to hold the "shift" key