Matrix A_p weights via S-functions

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Preliminary version

0. Introduction

In [TV1] the matrix A_2 condition appeared. Later F. Nazarov came up with the conjecture what matrix A_p should look like. This conjecture turned out to be correct and the Bellman function approach to weighted matrix inequalities for singular integrals was developed in [NT].

Here we suggest another approach to proving matrix weighted inequalities.

Notations. Symbol f_p^{02} denotes the sequences $s = \{s_I\}_{I\in D}$ enumerated by the set of dyadic intervals D such that

$$\|s\|^p_{f_p^{02}} = \int \left(\sum_{I\ni x} |s_I|^2 \frac{1}{|I|}\right)^{p/2} dx < \infty.$$

Similarly one can introduce $f_p^{02}(\mathbb{C}^d)$ just replacing numbers s_I by $\mathbb{C}^d$ vectors s_I and $|\cdot|$ by $\|\cdot\|$. One can consider $f_p^{02}(w)$, which is a weighted Triebel-Lizorkin space. This amounts to changing the integration dx to $w(x)dx$. As to the vector weighted norm one just replaces $\|s_I\|^2$ by $\|W^{1/p}(x)s_I\|^2$ to obtain the definition of $f_p^{02}(\mathbb{C}^d, W)$, where $W(x)$ is now positive self-adjoint operator (matrix).

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1 Matrix A_p condition

Let $\| \ \|$ denote the standard Hilbert norm in $\mathbb{C}^d$, and let $(\ , \ )$ be the scalar product in $\mathbb{C}^d$. If ρ denotes a norm in $\mathbb{C}^d$ we can notice that $\rho(x) \underset{d}{\asymp} \|Ax\|$ where A is a positive self-adjoint operator and constants depend only on the dimension d. In fact, let us consider the ellipsoid of the largest volume contained in the ρ-unit ball $\{x : \rho(x) \leq 1\}$, which is a convex subset of $\mathbb{C}^d$. Clearly, $\{x : \|Ax\| \leq 1\} \subset \{x : \rho(x) \leq 1\} \subset C_d\{x : \|Ax\| \leq 1\}$, where A is a positive matrix mapping the ellipsoid onto the standard Hilbert ball.

If we have a family ρ_t of norms and consider $L^p(\rho) = \{f : (\int \rho_t(f)^p dt)^{1/p} < \infty\}$ we can also introduce an equivalent norm $\|f\|_{L^p(\rho)} \asymp (\int \|A(t)f\|^p dt)^{1/p}$, with $A(t)$ as above constructed for ρ_t.

In weighted matrix inequalities one deals usually with $L^p(W)$ spaces, where $W(t)$ is matrix function with positive (a.e) $W(t) : \mathbb{C}^d \to \mathbb{C}^d$, and $\|f\|_{L^p(W)} \stackrel{def}{=} (\int \|W(t)^{1/p} f\|^p)^{1/p}$.

As we see one can consider $L^p(\rho)$ as well and reduce it to $L^p(W)$ by setting $A(t) = W^{1/p}(t)$.

Sometimes it is more convenient to use language of ρ and sometimes that of W.

Remind that the dual metric ρ^* is given by

$$\rho^*(x) = \sup_{y \neq 0} \frac{|(x,y)|}{\rho(y)}.$$

We can identify $(L^p(\rho))^*$ with $L^q(\rho^*)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $(L^p(W))^*$ with $L^q(U)$, where $W^{1/p}$-metric should be dual to $U^{1/q}$-metric, that is

$$W^{1/p}(t) = U^{-1/q}(t), \quad \text{for a.e.} \quad t$$

To introduce Nazarov's analog of A_p for matrix weights we use the language of metrics ρ and their averagings. Consider the norms (I is any interval)

$$\rho_{p,I}(x) \stackrel{def}{=} \left(\frac{1}{I} \int_I \rho_t^p(x) dt \right)^{1/p}$$

The unit ball again can be "reduced" to an ellipsoid, that is there exists a positive A_I:

$$\rho_{p,I}(x) \underset{d}{\asymp} \|A_I x\|.$$

Let us reduce the dual metric as well:

$$\rho^*_{q,I}(x) \underset{d}{\asymp} \|B_I x\|.$$

Proposition 1.1.

1) $|(x,y)| \le \rho_{p,I}(x)\rho^*_{q,I}(y)$

2) $(\rho_{p,I})^* \le \rho^*_{q,I}$.

Proof. 1) Obvious:

$$|(x,y)| = \frac{1}{I}\int_I |(x,y)| \le \frac{1}{I}\int_I \rho_t(x)\rho^*_t(y) \le \rho_{p,I}(x)\rho^*_{q,I}(y).$$

2) It is a paraphrase of the first assertion. ■

Definition. If we have an opposite inequality

$$\rho^*_{q,I} \le C(\rho_{p,I})^*, \quad \forall\, I$$

then ρ is called an A_p-metric. If $\rho_t(x) = \|W^{1/p}(t)x\|$ and ρ is an A_p-metric then W is called an A_p (matrix) weight. ■

Proposition 1.2. The following assertions are equivalent

1) ρ is an A_p-metric;

2) $\rho_{p,I} \le C(\rho^*_{q,I})^*$;

3) if $\rho_{p,I}(x) \asymp \|A_I x\|$, $\rho^*_{q,I}(x) \asymp \|B_I x\|$

then $B_I^2 \asymp A_I^{-2}$ in the sense of self adjoint operators. ■

Proof. Obvious. ■

Remark. For $p = q = 2$ the operators A_I and B_I can be calculated if we use the language of matrix weight W.

$$A_I = (W_I)^{1/2},\ B_I = (W^{-1})_I^{1/2}.$$

It is easy to see that always

$$(W_I)^{-1} \le (W^{-1})_I.$$

The A_2-condition obtained in [TV1], [TV2] had a form

$$(W^{-1})_I \le C(W_I),$$

where inequality is understood in the sense of self-adjoint operators.

2 Matrix A_∞ condition

Let us introduce $A_{p,\infty}$ condition on metric which serves as the analog of A_∞ condition for scalar weights. Its definition goes exactly along the lines of definition of A_p. Only $\rho^*_{q,I}$ is replaced by

$$\rho^*_{o,I}(x) \stackrel{def}{=} e^{\frac{1}{I}\int_I \log \rho^*_t(x)dt}.$$

Proposition 2.1. $(\rho_{p,I})^* \le \rho^*_{o,I}$.

Proof. The inequality can be rewritten as follows

$$(\rho^*_{o,I})^* \le \rho_{p,I}.$$

Then it follows from two elementary observations:

$$\rho_{o,I} \le \rho_{p,I}$$

which is just Jensen's inequality. And

$$(\rho^*_{o,I})^* \le \rho_{o,I}. \tag{2.1}$$

This last inequality follows from

$$\log |(x,y)| - \frac{1}{I}\int_I \log \rho^*_t(y)dt \le \frac{1}{I}\int_I \log \rho_t(x)dt.$$

And this is the integration of the logarithm of $|(x,y)| \le \rho^*_t(y)\rho_t(x)$. ■

Definition. If we have an opposite inequality

$$\rho^*_{o,I} \le C(\rho_{p,I})^*, \quad \forall\, I \tag{2.2}$$

which is equivalent to

$$\rho_{p,I} \le C(\rho^*_{o,I})^*, \quad \forall\, I \tag{2.3}$$

then ρ is called $A_{p,\infty}$-metric. If $\rho_t(x) = \|W^{1/p}(t)x\|$ and ρ is an $A_{p,\infty}$-metric then W is called an $A_{p,\infty}$ matrix weight. ■

Remark. By Jensen's inequality $\rho^*_{o,I} \le \rho^*_{q,I}$ and thus $A_p \Rightarrow A_{p,\infty}$. ■

Let B_ρ denote the unit ball in the metric ρ.

Proposition 2.2. The following assertions are equivalent.

1) $\rho \in A_{p,\infty}$;

2) $\frac{1}{I}\int_I \log \mathrm{vol}\ B_{\rho_t} dt \le C + \log \mathrm{vol}\ B_{\rho_{p,I}}$ for all I.

In particular, $W \in A_{p,\infty}$ if and only if

$$\det A_I \le C e^{\frac{1}{I}\int_I \log\det W(t)^{1/p} dt}, \quad \forall\, I,$$

where A_I is a positive definite operator reducing $\rho_{p,I}$ in the previous sense, namely

$$\rho_{p,I}(x) \asymp \|A_I x\| \quad \blacksquare$$

Proof. 1) $\Rightarrow$ 2). It is convenient to use the language of matrices. So let $\rho_t(x) \asymp \|A(t)x\|$, $\rho_{p,I}(x) \asymp \|A_I x\|$. let $\{e_i\}_{i=1}^d$ be an orthonormal basis of eigenvectors of A_I. We apply 1) to each of these vectors:

$$\frac{1}{I}\int_I \log \|A(t)^{-1} e_i\| \le C + \log \|A_I^{-1} e_i\|, \quad i = 1, \ldots, d.$$

Use the following elementary observation; for any positive definite $d \times d$ matrix Q and any orthonormal basis $\{e_i\}_{i=1}^d$ in $\mathbb{C}^d$, $\det Q \le \prod_{i=1}^d (Qe_i, e_i) \le \prod_{i=1}^d \|Qe_i\|$. This inequality is just the consequence of arithmetic mean-geometric mean inequality. Summing up the inequalities above we get

$$\frac{1}{I}\int_I \log\det\ A(t)^{-1} dt \le C + \log\det\ A_I^{-1}$$

which means

$$\frac{1}{I}\int_I \log \mathrm{vol}\ B_{\rho_t} \le C + \log \mathrm{vol}\ B_{\rho_{p,I}}.$$

2) $\Rightarrow$ 1). Let g be a unit vector. Choose $g_1(t) \equiv g,\ g_2(t), \ldots, g_d(t)$ to be an orthonormal basis such that for $C(t) \stackrel{def}{=} A_I^{-1} A(t)$ we have $\det C(t)^{-1} \ge C_d \prod_{i=1}^d \|C(t)^{-1} g_i(t)\|$. We know that

$$\begin{aligned}
\frac{1}{I}\int_I \|C(t) g_i(t)\|^p &\le C_{d,p} \sum_{j=1}^d \frac{1}{I}\int_I |(C^*(t)e_j, g_j(t)|^p \\
&\le C_{d,p} \sum_{j=1}^d \|A(t) A_I^{-1} e_j\|^p \asymp C_{d,p} \sum_{j=1}^d \rho_{p,I}^p (A_I^{-1} e_j) \\
&< C'_{d,p} \le \infty.
\end{aligned}$$

In particular,

$$\sum_{i=1}^{d} \frac{1}{I} \int_I \log^+ \|C(t) g_i(t)\| dt \le \frac{1}{p} C'_{d,p}.$$

We are given $\frac{1}{I} \int_I \log \det\, C(t)^{-1} dt \le C$ which implies

$$\begin{aligned}
\frac{1}{I} \int_I \log^+ \|C(t)^{-1} g\| dt &\le C + \sum_{i=2}^{d} \frac{1}{I} \int_I \log^- \|C(t)^{-1} g_i(t)\| \\
&\le C + \sum_{i=2}^{d} \frac{1}{I} \int_I \log^+ \|C(t) g_i(t)\| dt \\
&\le C + \frac{1}{p} C'_{d,p} = C'' < \infty.
\end{aligned}$$

This implies

$$\frac{1}{I} \int_I \log \|A(t)^{-1} g\| dt \le C'' + \log \|A_I^{-1} g\|. \quad \blacksquare$$

3 Embedding theorem

In this section we deal with $\rho \in A_{p,\infty}$. We consider the "reducing" self-adjoint operators A_I:

$$\left(\frac{1}{I} \int_I \rho_t(x)^p dt \right)^{1/p} \asymp \|A_I x\|. \tag{3.1}$$

Then $A_{p,\infty}$ condition can be written as $(\rho_{p,I})^* \ge c \rho^*_{o,I}$, that is

$$\forall\ J,\ \frac{1}{J} \int_J \log \rho_t^*(x) dt \le C + \log \|A_J^{-1} x\|, \tag{3.2}$$

with inverse inequality being always valid as well with $C = 0$ (see Proposition 2.1).

Lemma 3.1. *Let $\rho \in A_{p,\infty}$, and given a dyadic I let $\mathcal{F}_k$ denote the family of dyadic J, $J \subset I$, such that*

$$\|A_J^{-1} A_I\| \ge e^k.$$

Then

$$|\bigcup_{J\in\mathcal{F}_k} J| \leq \frac{c}{k}|I|. \quad \blacksquare \tag{3.3}$$

Proof. Let $\{e_i\}_{i=1}^d$ be a standard basis in $\mathbb{C}^d$. Denote $\mathcal{F}_k^i$ the set of dyadic intervals with

$$\|A_J^{-1}A_I e_i\| \geq e^k.$$

First use (3.2) to conclude that

$$\frac{1}{I}\int_I \log \rho_t^*(A_I e_i) \leq C. \tag{3.4}$$

Then use the inverse to (3.2) to conclude that for $J \in \mathcal{F}_k^i$

$$\frac{1}{J}\int_J \log^+ \rho_t^*(A_I, e_i) \geq \log \|A_J^{-1}A_I e_i\| \geq k. \tag{3.5}$$

If we could replace log by $\log^+$ in (3.4) we would be done by standard argument. To do this replacement let us notice that

$$\frac{1}{I}\int_I \log^+ \rho_t^*(A_I e_i) = \frac{1}{I}\int_I \log \rho_t^*(A_I e_i) + \frac{1}{I}\int_I \log^+ \frac{1}{\rho_t^*(A_I e_i)}$$
$$\leq C + \frac{1}{pI}\int_I \log^+ \frac{1}{(\rho_t^*(A_I e_i))^p} \leq C + \frac{1}{pI}\int_I \log^+(\rho_t(A_I^{-1}e_i))^p$$
$$\leq C + \frac{1}{pI}\int_I \rho_t^p(A_I^{-1}e_i) \leq C + \frac{1}{p}(C\|e_i\|)^p \leq C'.$$

In this chain we used an elementary inequality

$$\rho_t(A_I^{-1}e_i)\rho_t^*(A_I e_i) \geq |(e_i, e_i)| = 1.$$

The lemma is proved. $\blacksquare$

Remark. It is needless to say that for the scalar situation all $A_{p,\infty}$ just coincide with A_∞. Lemma 3.1 reflects a characteristic property of A_∞ functions: the set where the function is much smaller than its average is small. $\blacksquare$

Let h_J denote a standard normalized in $L^2(dt)$ Haar function.

Before stating the theorem we need to introduce a new type of weighted Triebel-Lizorkin spaces. Given a family of linear operators $\{C_J\}_{J\in D}$ we consider $f_p^{02}(\mathbb{C}^d, \{C_J\})$ to consist of sequences $\{s\}_{J\in D}$ such that $\{C_J s_J\} \in f_p^{02}$.

Here is the embedding theorem which gives the name to the section. As always $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 3.2 *Let $\rho \in A_{p,\infty}$ and let $\{A_J\}$ be its reducing operators as in (3.1). Let $s_J = \int_J g\, h_J$. Then the following inequality holds*

$$\|\{s_J\}\|_{f_q^{02}(\{A_J^{-1}\})} \leq C\|g\|_{L^q(\rho^*)}. \quad \blacksquare$$

Remarks. 1) Even in the simplest case $p = q = 2$ and the scalar weight $w \in A_\infty$ this result seems to be rather new (although we did not check all the literature). It amounts to the inequality

$$w \in A_\infty \Rightarrow \sum_{J\in D} (w_J)^{-1} |(g, h_J)|^2 \leq C\|g\|^2_{L^2(\frac{1}{w})},$$

which was mentioned in [TV1], [TVZ]. Notice that when $w \in A_2$ the left part is equivalent to

$$\begin{aligned} \sum (w^{-1})_J |(g, h_J)|^2 &= \int \sum_{J\ni x} |(g,h_J)|^2 \frac{1}{|J|} w^{-1}(x)dx \\ &= \|\{(g,h_J)\}\|^2_{f_2^{02}(w^{-1})} \asymp \|g\|^2_{L^2(w^{-1})}, \end{aligned}$$

2) We do not know whether a stronger inequality

$$w \in A_\infty \Rightarrow \sum_{J\in D} (w^{-1})_J |(g, h_J)|^2 \leq C\|g\|^2_{L^2(\frac{1}{w})}$$

holds, which would mean exactly that for $w \in A_\infty$ the operator $g \to \{(g, h_J)\}_{J\in D}$ maps $L^2(w^{-1})$ into $f_2^{02}(w^{-1})$.

3) There are interesting general relationships between

$$f_p^{02}(\mathbb{C}^d, \rho) \stackrel{def}{=} \left\{ \{s_I\} : \int \left(\sum_{J\ni t} \rho_t^2(s_I) \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}$$

and

$$f_p^{02}(\mathbb{C}^d, \{C_I\}) = \left\{ \{s_I\} : \int \left(\sum_{I \ni t} \|C_I s_I\|^2 \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}.$$

Quite naturally one should choose C_I reducing ρ_t:

$$\left(\frac{1}{I} \int_I \rho_t^p(x) dt \right)^{1/p} \asymp \|C_I x\| \tag{3.6}$$

and then

$$f_p^{02}(\mathbb{C}^d, \{C_I\}) = \left\{ \{s_I\} : \int \left(\sum_{I \ni t} \left(\frac{1}{I} \int_I \rho_s^p(s_I) ds \right)^{2/p} \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}.$$

It is clear that for $p = 2$ these spaces coincide. It will be proved elsewhere that for $p \geq 2$, $\rho \in A_{p,\infty}$ they also coincide. Clearly, many questions appear: what about $p < 2$?, what if we change exponent p to another one in (3.6)?, etc...

Notice that if $W \in A_p$ these spaces coincide.

4) The space $f_q^{02}(\mathbb{C}^d, \{A_J^{-1}\})$ is not of the type considered at 3). The difference is that now $C_J = A_J^{-1}$ are not generated as reducing operators as it was in (3.6). They are inverses to reducing operators.

Notice that if $W \in A_p$ then inverses to reducing operators are also reducing for the dual weight $U = W^{-q/p}$ and in this particular case we are going to prove in the next section that

$$L^p(W) \approx f_p^{02}(\mathbb{C}^d, \{A_J\}), \tag{3.7}$$

where A_J are reducing operators introduced in (3.1). ■

Proof of Theorem 3.2. Fix $g \in L^q(\rho^*)$ and a test function $f \in L^p$. We need to show that

$$\sum_{J \in D} \left| \left(\int g\, h_J, A_J^{-1} \int f\, h_J \right) \right| \leq C \|f\|_{L^p} \|g\|_{L^q(\rho^*)}. \tag{3.8}$$

The sum on the left can be rewritten as

$$\int \left(\sum_{J \ni t} \left| \left(\int g\, h_J, A_J^{-1} \int f\, h_J \right) \right| \frac{1}{|J|} \right) dt \stackrel{def}{=} \int S(t) dt$$

Let us consider a nonnegative function $h(t)$ and introduce

$$S_{h(t)}(t) = \sum_{\substack{J \ni t \\ |J| \leq h(t)}} \left| \left(\int g\, h_J, A_J^{-1} \int f\, h_J \right) \right| \frac{1}{|J|}.$$

$S_{h(t)}(t) \leq S(t)$ but

$$\int S_{h(t)}(t)dt \geq a \int S(t)dt$$

if the function $h(t)$ has the following property:

$$\forall\, J \in D \quad |\{t \in J : h(t) \geq |J|\}| \geq a|J|. \tag{3.9}$$

We are going to choose now a function h satisfying (3.9) and such that

$$S_{h(t)}(t) \leq B \left(M\|f\|^{p_*}\right)^{1/p_*}(t)\ \left(M\left(\rho^*(g)\right)^{q_*}\right)^{1/q_*}(t) \tag{3.10}$$

for certain $B < \infty$, $1 < p_* < p$, $1 < q_* < q$. M denotes the maximal function. After (3.10) we are done:

$$\begin{aligned}
\int S(t)dt &\leq a^{-1} \int S_{h(t)}(t)dt \leq a^{-1} \int \left(M\|f\|^{p_*}\right)^{1/p_*}(t) \cdot \left(M\rho^*(g)^{q_*}\right)^{1/q_*}(t) \\
&\leq a^{-1} \left(\int \left(M\|f\|^{p_*}\right)^{p/p_*} \right)^{1/p} \left(\int \left(M\rho^*(g)^{q_*}\right)^{q/q_*} \right)^{1/q} \\
&\leq C(a, p/p_*, q/q_*) \left(\int \|f\|^p \right)^{1/p} \left(\int \rho^*(g)^q \right)^{1/q} \\
&= C(a, p/p_*, q/q_*) \|f\|_{L^p} \cdot \|g\|_{L^q}(\rho^*),
\end{aligned}$$

which is what we need.

To choose $h(t)$ satisfying (3.9) and (3.10) we follow the algorithm below.

Fix a dyadic interval I. Fix a large k and delete from I all the dyadic intervals for which

$$\|A_I A_J^{-1}\| \geq e^k.$$

By Lemma 3.1 the set E_I which rests has the property that

$$|E_I| \geq \frac{1}{2}|I|.$$

Let us introduce

$$t \in I, \qquad S^{A_I^{-1}}(g)(t) = \left(\sum_{\substack{J \ni t \\ J \subseteq I}} \|A_I^{-1} \int g\, h_J\|^2 \frac{1}{|J|} \right)^{1/2};$$

$$t \in I, \qquad S^{A_I A_J^{-1}}(f)(t) = \left(\sum_{\substack{J \ni t, \\ J \subseteq I}} \|A_I A_J^{-1} \int f\, h_J\|^2 \frac{1}{|J|} \right)^{1/2}.$$

We are interested in proving that there are relatively large subsets of I on which these S-functions are smaller than maximal functions from (3.10).

To find such sets it is sufficient to estimate any averages of these S-functions. Fix $\alpha > 1$ and very close to 1.

$$\begin{aligned} \frac{1}{I} \int_I \left(S^{A_I^{-1}}(g)(t) \right)^{\alpha} dt &\leq C(\alpha) \frac{1}{I} \int \|A_I^{-1} g \cdot \chi_I\|^{\alpha} dt \\ &= C(\alpha) \frac{1}{I} \int_I \|A_I^{-1} g\|^{\alpha} dt. \end{aligned}$$

The first inequality follows from the classical L^{α} estimate of S-function by L^{α} norm of the function when $\alpha > 1$ (see e.g. [St]). To continue the estimate we use that we are in the space of finite dimension to write ($\{e_i\}$ is a standard basis of $\mathbb{C}^d$):

$$\|A_I^{-1} g(t)\| \leq C_d \sum_{i=1}^{d} |(A_I^{-1} g(t), e_i)| \leq C_d \sum_{i=1}^{d} \rho_t^*(g) \rho_t(A_I^{-1} e_i)$$

Thus

$$\begin{aligned} \|S^{A_I^{-1}}(g)\|_{L^{\alpha}(\frac{dt}{I})} &\leq C(\alpha, d) \sum_{i=1}^{d} \left(\frac{1}{I} \int_I \rho_t^{(p+\varepsilon)\alpha}(A_I^{-1} e_i) \right)^{\frac{1}{(p+\varepsilon)\alpha}} \\ &\quad \cdot \left(\frac{1}{I} \int_I \rho_t^*(g)^{(q-\delta)\alpha} \right)^{\frac{1}{(q-\delta)\alpha}} \qquad (3.11) \end{aligned}$$

Here $(p+\varepsilon)^{-1} + (q-\delta)^{-1} = 1$, and ε is chosen to be very small to guarantee $q - \delta > 1$. The choice of ε and α is also dictated by the observation that the positive function $t \to \rho_t^p(x)$ is an A_∞ function if $\rho \in A_{p,\infty}$.

In fact, inequalities (2.1) and (2.3) imply that

$$\frac{1}{I}\int_I \rho_t^p(x)dt \le C e^{\frac{1}{I}\int_I \log \rho_t^p(x)dt} \tag{3.12}$$

which is equivalent to A_∞ (see [GCRF]). As $\rho_t^p(x)$ is (uniformly in x) in A_∞ it satisfies (uniformly in x) a reverse Hölder inequality. In particular, there exist a $\bar{p} > p$ such that

$$\forall\, I,\ \forall\, x,\ \left(\frac{1}{I}\int_I \rho_t^{\bar{p}}(x)dt\right)^{\frac{1}{\bar{p}}} \le C\left(\frac{1}{I}\int_I \rho_t^p(x)dt\right)^{1/p} \tag{3.13}$$

Now we choose α close to 1 and ε close to 0 to guarantee that

$$(p+\varepsilon)\alpha \le \bar{p}.$$

In particular, (3.13) implies

$$\begin{aligned}\left(\frac{1}{I}\int_I \rho_t^{(p+\varepsilon)\alpha}(A_I^{-1}e_i)dt\right)^{\frac{1}{(p+\varepsilon)\alpha}} &\le C\left(\frac{1}{I}\int_I \rho_t^p(A_I^{-1}e_i)\right)^{1/p}\\ &\le C'\|A_I A_I^{-1}e_i\| = C',\end{aligned} \tag{3.14}$$

where we use the definition (3.1) of A_I.

Now we can continue (3.11) using (3.14):

$$\|S^{A_I^{-1}}(g)\|_{L^\alpha(I,\frac{dt}{I})} \le C(\alpha,d)\cdot d\cdot C'\cdot \inf_{t\in I}\left[M(p^*(g)^{q_*})(t)\right]^{\frac{1}{q_*}}, \tag{3.15}$$

where $q_* = (q-\delta)\cdot\alpha \in (1,q)$ if $\delta = \delta(\varepsilon)$ and α are chosen properly.

This last inequalities ensures that for any τ, $\tau \in (0,1)$, we can find the subset $E(\tau,I) \subset I$, $|E(\tau,I)| \ge (1-\tau^\alpha)|I|$ such that

$$t\in E(\tau,I) \Rightarrow S^{A_I^{-1}}(g)(t) \le \frac{C''(\alpha,d)}{\tau}\inf_{t\in I}\left[M(\rho^*(g)^{q_*})\right]^{1/q_*} \tag{3.16}$$

This is the main estimate of our first S-function.

Now we are going to give a similar estimate for the second S-function $S^{A_I A_J^{-1}}(f)(t)$. Again choose $\alpha > 1$ and close to 1, and we are going to estimate $\frac{1}{|E_I|}\int_{E_I}\left(S^{A_IA_J^{-1}}(f)(t)\right)^\alpha dt$. For $t\in E_I$ all $J\ni t, J\subset I$ have the property

$$\|A_IA_J^{-1}\|\le e^k$$

and so

$$t\in E_I \quad S^{A_IA_J^{-1}}(f)(t)\le S^I(f)(t)\stackrel{def}{=}\left(\sum_{\substack{J\ni t\\ J\subset I}}\|\int f\,h_J\|^2\frac{1}{|J|}\right)^{1/2}.$$

Thus

$$\frac{1}{|E_I|}\int_{E_I}\left(S^{A_IA_J^{-1}}(f)(t)\right)^\alpha dt\le\frac{2}{I}\int_I\left(S^I(f)(t)\right)^\alpha dt\le C(\alpha)\frac{1}{I}\int_I\|f\|^\alpha dt.$$

Choosing $1<\alpha<p$ and denoting it by p_* we get

$$\|S^{A_IA_J^{-1}}(f)\|_{L^\alpha(E_I,\frac{dt}{|E_I|})}\le C(\alpha)\inf_{t\in I}(M\|f\|^{p_*})^{1/p_*}(t).$$

As before it implies for any $\tau\in(0,1)$ the existence of $e(\tau,I)\subset E_I$, $|e(\tau,I)|\ge(1-\tau^\alpha)|E_I|\ge\frac{1}{2}(1-\tau^\alpha)|I|$ such that

$$t\in e(\tau,I)\Rightarrow S^{A_IA_J^{-1}}(f)(t)\le\frac{C(\alpha)}{\tau}\inf_{t\in I}(M\|f\|^{p_*})^{1/p_*}. \tag{3.17}$$

Choose now τ so small that

$$|e(\tau,I)\cap E(\tau,I)|\ge\frac{1}{4}|I|. \tag{3.18}$$

We are ready to choose the height function $h(t)$ satisfying (3.9) and (3.10). Put

$$\begin{aligned} h(t) &\stackrel{def}{=} \sup\{h: S_h(t)\stackrel{def}{=}\sum_{\substack{J\ni t\\ |J|\le h}}\frac{1}{|J|}\left|\left(\int g\,h_J, A_J^{-1}\int f\,h_J\right)\right| \\ &\le B(M\|f\|^{p_*})^{1/p_*}(t)(M\rho^*(g)^{q_*}(t)\}, \end{aligned} \tag{3.19}$$

where p_*, q_* appeared in (3.17) and (3.16) and where $B=\frac{C''(\alpha,d)C(\alpha)}{\tau^2}$, with constants taken from (3.16) and (3.17).

Then (3.10) is fulfilled automatically. To check (3.9) let us fix any $I \in D$ and let us consider for $t \in I$

$$S_I(t) \stackrel{def}{=} \sum_{\substack{J \ni t \\ J \subseteq I}} \frac{1}{|J|} \left| \left(\int g\, h_J, A_J^{-1} \int f\, h_J \right) \right| .$$

Notice that

$$\begin{aligned} S_I(t) &= \sum_{\substack{J \ni t \\ J \subseteq I}} \left| \left(A_I^{-1} \int g\, h_J, A_I A_J^{-1} \int f\, h_J \right) \right| \\ &\leq S^{A_I^{-1}}(g)(t) S^{A_I A_J^{-1}}(f)(t). \end{aligned}$$

Thus (3.16) and (3.17) applied for $t \in e(\tau, I) \cap E(\tau, I)$ show that on a least a quarter of I (see (3.18))

$$S_I(t) \leq B(M\|f\|^{p_*})^{1/p_*}(t)(M\rho^*(g)^{q_*})^{1/q_*}(t).$$

Thus on a quarter of I the function h just defined in (3.19) is greater than $|I|$. Because I was arbitrary, (3.9) holds and Theorem 3.2 is finished. ∎

4 $T_\pm$ theorem

Theorem 3.2 means that the operator $\sum A_J^{-1} h_J \otimes h_J$ given by

$$g \to \sum_{J \in D} A_J^{-1} \left(\int g\, h_J \right) \cdot h_J$$

is bounded $L^q(\rho^*) \to f_q^{02}$. If we use the language of matrix functions, meaning that $\rho_t(x) = \|W^{1/p}(t)x\|$, $\rho_t^*(x) = \|U^{1/q}(t)x\|$, where $U^{1/q} = W^{-1/p}$ we can make a change of variable $g(t) = W^{1/p}(t)G(t)$ now having $G \in L^q$ with no weight. The operator now can be written in the form $\sum A_J^{-1} h_J \otimes W^{1/p} h_J$. If A_J is given by (3.1) the sum above is proved to be a bounded operator from L^q to f_q^{02}. The same proof certainly shows that operators $\sum \pm A_J^{-1} h_J \otimes W^{1/p} h_J$ are uniformly bounded.

All these claims were true under the condition that ρ (or W) belongs to $A_{p,\infty}$.

Now if ρ (or W) belongs to A_p one can say more. We summarize it in two theorems.

Theorem 4.1. *Let $\rho \in A_p$. Then if A_J are from (3.1) and $s_J = \int f\, h_J$ we have*

$$\|\{s_J\}\|_{f_p^{02}(\{A_J\})} \asymp \|f\|_{L^p(\rho)}. \quad \blacksquare \tag{4.1}$$

Proof. It follows immediately from the definition that $\rho \in A_p \Rightarrow \rho \in A_{p,\infty}$ and $\rho \in A_p \Rightarrow \rho^* \in A_q \Rightarrow \rho^* \in A_{q,\infty}$.

Let us also introduce the reducing operators for ρ^*:

$$\left(\frac{1}{J}\int_J \rho_t^*(x)^q dt\right)^{1/q} \asymp \|C_J x\|.$$

Let us apply Theorem 3.2 to $\rho^* \in A_{q,\infty}$ to obtain

$$\left\|\left\{\int f\, h_J\right\}\right\|_{f_p^{02}(\{C_J^{-1}\})} \leq C\|f\|_{L^p(\rho)}.$$

Notice that $\rho \in A_p$ means that $C_J^{-2} \asymp A_J^2$ and so we get

$$\left\|\left\{\int f\, h_J\right\}\right\|_{F_p^{02}(\{A_J\})} \leq C\|f\|_{L^p(\rho)}.$$

To obtain the inverse inequality we use duality and again Theorem 3.2 for $\rho \in A_{p,\infty}$ to obtain

$$\begin{aligned}
\|f\|_{L^p(\rho)} &= \sup_{\|g\|_{L^q(\rho^*)}\leq 1} |\langle f, g\rangle| = \sup\left|\sum\left(\int f\, h_J, \int g\, h_J\right)\right| \\
&= \sup\left|\sum\left(A_J\int f\, h_J, A_J^{-1}\int g\, h_J\right)\right| \\
&\leq \left\|\left\{\int f\, h_J\right\}\right\|_{f_p^{02}(\{A_J\})} \cdot \left\|\left\{\int g\, h_J\right\}\right\|_{f_q^{02}(\{A_J^{-1}\})} \\
&\leq C\left\|\left\{\int f\, h_J\right\}\right\|_{f_p^{02}(\{A_J\})}.
\end{aligned}$$

The last inequality is Theorem 3.2. We are done. $\blacksquare$

Remark. It seems to us that these theorem gives the new information about $L^p(w), w \in A_p$, even in the sclar case. The reader should notice that we organized a canonical isomorphism of $L^p(w)$ and f_p^{02}. This is not the classical isomorphism $f \to \{(f, h_J)\}_{J \in D}$ between $L^p(w)$ and $f_p^{02}(w)$.

Corollary 4.2. *Operators*

$$T_\pm \stackrel{def}{=} \sum \pm h_J \otimes h_J$$

defined by the formula

$$T_\pm \varphi = \sum_{J \in D} \pm h_J \cdot \int \varphi h_J$$

are uniformly bounded in $L^p(\rho)$ if and only if $\rho \in A_p$. $\blacksquare$

Proof. "If" part follows immediately from Theorem 4.1. In fact, we just use the equivalent norm $\|\cdot\|_{f_p^{02}(\{A_J\})}$ in $L^p(\rho)$ which does not change if φ is replaced by $T_\pm\varphi$ because $\int T_\pm \varphi \cdot h_J = \pm \int \varphi\, h_J$. "Only if" part follows from the uniform boundedness of rank one operators $h_J \otimes h_J$. It follows exactly as the first part of Theorem 5.1 below. So we send the reader to this coming proof. $\blacksquare$

The next result is an analog of classical isomorphism $L^p(w) \approx f_p^{02}(w)$.

Theorem 4.3. *Let $\rho \in A_p$. Let $s_J = \int f\, h_J$. Then*

$$\|\{s_J\}\|_{f_p^{02}(\rho)} \asymp \|f\|_{L^p(\rho)}. \quad \blacksquare \tag{4.2}$$

Proof. It follows easily from the corollary. In fact, we know that

$$\|f\|^p_{L^p(\rho)} \asymp \int \rho_t^p \left(\sum_{J \in D} \pm h_J \cdot \int f\, h_J \right) dt$$

We integrate over the signs and use the Khintchin theorem which is valid here. We obtain

$$\begin{aligned} \|f\|^p_{L^p(\rho)} &\asymp \int \left(\sum_{J \ni t} \rho_t^2 (\int f\, h_J) \frac{1}{|J|} \right)^{p/2} dt \\ &= \|\{s_J\}\|^p_{f_p^{02}(\rho)}. \quad \blacksquare \end{aligned}$$

Remark. In particular, if $\rho \in A_p$, and A_J are reducing operators for ρ considered in (3.1) we see from (4.1) and (4.2) the following equivalence of norms in sequence spaces

$$\|\{s_J\}\|_{f_p^{02}(\rho)} \asymp \|\{s_J\}\|_{f_p^{02}(\{A_J\})}. \tag{4.3}$$

But it has been already mentioned in the remarks to Theorem 3.2 that this equivalence does not require $\rho \in A_p$. For $p = 2$ it holds always. For $p \geq 2$, $\rho \in A_{p,\infty}$ would be sufficient for (4.3). This can be proved along the lines of Theorem 3.2.

We would like to ask for the criterion for (4.3) to hold.

5 Hilbert transform

Theorem 4.1 and Corollary 4.2 allow one to show that any Calderon-Zygmund operator with $T1 = T^*1 = 0$ is bounded in $L^p(\rho)$ if $\rho \in A_p$. But still the reduction of any T to $T_\pm$ is very technical and difficult task. In this section we show that at least for the Hilbert transform one can give a relatively easy and straightforward proof very similar to one of Theorem 3.1.

Theorem 5.1. *Let $\rho \in A_p$. Then*

$$\|P_+ f\|_{L^p(\rho)} \leq C\|f\|_{L^p(\rho)}. \tag{5.1}$$

Conversely if (5.1) holds then $\rho \in A_p$. ■

Before proving Theorem 5.1 let us prove two simple lemmas. Consider the Poisson kernel $P_I = \dfrac{|I|}{x^2 + |I|^2}$ and define $k_{I,q} = P_I^{q/2} \cdot |I|^{q/2-1}$ $\quad (1 < q < \infty)$. Then $\int k_{I,q} \asymp 1$ and here is the first lemma.

Lemma 5.2. *If u is a scalar A_q weight then $\int u k_{I,q} \leq C \dfrac{1}{|I|} \displaystyle\int_I u$.* ■

Proof. We can write $k_{I,q} \asymp \sum_{k=1}^{\infty} \varepsilon_k \dfrac{1}{|2^k I|} \chi_{2^k I}$, where $\varepsilon_k \asymp (2^{-k})^{q/p}$. Now if $u \in A_q$ let $w = u^{-p/q} \in A_p$ and we can write the following chain of inequalities

$$
\begin{aligned}
(w_I)^{q/p}(\int k_{I,q}u) &\leq \sum_{k=1}^{\infty}(w_I)^{q/p}u_{2^kI}\varepsilon_k \\
&\leq \sum_{k=1}^{\infty}\frac{(w_I)^{q/p}}{(w_{2^kI})^{q/p}}\varepsilon_k = \sum_{k=1}^{\infty}\left(\frac{w(I)}{w(2^kI)}\right)^{q/p} 2^{k\cdot\frac{q}{p}}\varepsilon_k \\
&\leq C\sum_{k=1}^{\infty}\left(\frac{w(I)}{w(2^kI)}\right)^{q/p} \\
&\leq C\sum_{k=1}^{\infty}(1+\varepsilon)^{-k} \leq C'.
\end{aligned}
$$

The last inequality follows from the doubling property of A_p weights. We get $\int k_{I,q}u \leq C(w_I)^{-q/p} \leq C'u_I$ by using A_p condition again. ■

Lemma 5.3. *If W is a matrix A_p weight then $t \to \|W^{1/p}x\|^p$ is a scalar A_p weight for any vector x.* ■

Proof. It is convenient to use the language of $\rho_t(x) = \|W^{1/p}(t)x\|$. Fix x and choose $y(x)$ in such a way that $(\rho_{p,I})(x) = \sup_y \frac{|(x,y)|}{(\rho_{p,I})^*(y)} \leq 2\frac{|(x,y(x)|}{(\rho_{p,I})^*(y(x))}$. Now

$$
\begin{aligned}
\left(\frac{1}{|I|}\int_I \rho_t^p(x)\right)^{1/p}\left(\frac{1}{I}\int_I \rho_t^{-q}(x)\right)^{1/q} &\leq \rho_{p,I}(x)\left(\frac{1}{I}\int_I\left(\frac{\rho_t^*(y(x))}{|(x,y(x))|}\right)^q\right)^{1/q} \\
&\leq \frac{\rho_{p,I}(x)\rho_{q,I}^*(y(x))}{|(x,y(x))|} \\
&\leq C\frac{\rho_{p,I}(x)(\rho_{p,I})^*(y(x))}{|(x,y(x))|} \leq 2C. \quad ■
\end{aligned}
$$

Proof of Theorem 5.1. (5.1) $\Rightarrow \rho \in A_p$.

Operator P_+ has the kernel $\frac{1}{1-\bar{\zeta}z}$. We are given that it is a bounded operator. Fix an a in the unit disk and consider the operator with the kernel

$$\frac{1}{1-\bar{\zeta}z}\left[1-\left(\overline{\frac{\zeta-a}{1-\bar{a}\zeta}}\right)\frac{z-a}{1-\bar{a}z}\right].$$

Obviously it is a bounded operator in $L^p(\rho)$ too, actually it is given by the formula $P_+ - b_a P_+ \bar{b}_a$, where $b_a(\zeta) = \frac{\zeta - a}{1-\bar{a}z}$. Notice that pointwise multiplication by b_a does not change the norm in $L^p(\rho)$. Now let us make the simplification in the kernel to see that it equals to $\frac{1-|a|^2}{(1-\bar{\zeta}a)(1-\bar{a}z)} = k_a(z)\otimes k_a(\zeta)$, where $k_a(\zeta) = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}\zeta}$ is a normalized in L^2 Szegö kernel. Notice that

$$|k_a(\zeta)|^2 = P_a(\zeta),$$

where $P_a(\zeta)$ denotes the Poisson kernel.

Now let us use the fact (which we just derived from the boundedness of P_+) that $k_a\otimes k_a$ are uniformly bounded. Let $\varphi(t)\in L^q$, $\psi\in L^p$, and $\int \|\varphi\|^q dt \le 1$, $\int \|\psi\|^p dt \le 1$. Then

$$\left|\left(\int k_a(t)W^{1/p}(t)\varphi(t)dt, \int k_a(s)U^{1/q}(s)\psi(s)ds\right)\right| \le C$$

uniformly in a. Then denoting $k_a^{(p)} = (1-|a|^2)^{1/2-1/p}k_a$; $k_a^{(q)} = (1-|a|^2)^{1/2-1/q}k_a$, we have

$$\int |k_a^{(p)}(t)|^p \|W^{1/p}(t)\int k_a^{(q)}(s)U^{1/q}(s)\psi(s)ds\|^p \le C$$

Denoting $z = \int k_a^{(q)}(s)U^{1/q}(s)\psi(s)ds$ we conclude that

$$\left(\int |k_a^{(p)}|^p \|W^{1/p}(t)z\|^p\right)^{1/p} \le C\|\psi\|_{L^p}$$

This is an averaging as $\int |k_a^{(p)}|^p \asymp 1$. Denote it by $\rho_{p,a}$:

$$\rho_{p,a}(z) \le C\|\psi\|_{L^p}.$$

We want to show that

$$(\rho_{q,a}^*)^*(z) \geq C \inf\{\|\psi\|_{L^p} : z = \int k_a^{(q)}(s) U^{1/q}(s)\psi(s)\}.$$

But

$$\begin{aligned}
(\rho_{q,a}^*)^*(z) &= \max_y |(y,z)| / \left(\int \|U^{1/q}(s)y\|^q |k_a^{(q)}(s)|^q \right)^{1/q} \\
&= \max_y |\langle k_a^{(q)}(s) U^{1/q}(s) y, \psi(s)\rangle| / \|k_q^{(q)}(s) U^{1/q}(s) y\|_{L^q} \\
&= \inf\{\|\psi\|_{L^p} : z = \int k_q^{(q)}(s) U^{1/q}(s)\psi(s) ds\}.
\end{aligned}$$

Thus

$$\rho_{p,a}(z) \leq C(\rho_{q,a}^*)^*(z).$$

But $|k_a^{(p)}|^p \geq C(p)\dfrac{\chi_{I_a}}{|I_a|}$, where I_a is the interval of length $1-|a|$ centered at $a/|a|$. Thus

$$\rho_{p,I} \leq C(\rho_{q,I}^*)^*.$$

Now let us prove that

$\rho \in A_p \Rightarrow$ (5.1).

Let ρ is given by $W^{1/p}$ and ρ^* by $U^{1/q} = W^{-1/p}$. We have to prove that

$$f \to W^{1/p} P_+ W^{-1/p} f$$

is a bounded operator $L^p \to L^p$. Fix $g \in L^q$. We need to show

$$\int |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| \delta(z) dA(z) \leq C\|f\|_p \|g\|_q,$$

where $\delta(z)$ means the distance to the boundary. In other words, introducing a Stolz cones Γ_t, and

$$S(t) \stackrel{def}{=} \int_{\Gamma_t} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z)$$

one needs to prove that

$$\int S(t) dt \leq C\|f\|_p \|g\|_q \tag{5.2}$$

We follow closely the lines of the proof of Theorem 3.2. Let us consider a nonnegative function $h(t)$ and

$$S_{h(t)}(t) = \int_{\Gamma_{t,h(t)}} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z),$$

where

$$\Gamma_{t,h(t)} = \Gamma_t \cap \{z : \delta(z) \leq h(t)\}.$$

As in Section 3

$$\int S_{h(t)} dt \asymp \int S(t) dt \tag{5.3}$$

if the function $h(t)$ has the following property:

$$\forall\, J \quad |\{t \in J : h(t) \geq |J|\}| \geq a|J|. \tag{5.4}$$

Let us choose h to be maximal such that

$$S_{h(t)}(t) \leq B(M\|f\|^{p_*})^{1/p_*}(t)(M\|g\|^{q_*})^{1/q_*}(t) \tag{5.5}$$

where $B, p_* \in (1,p), q_* \in (1,q)$ will be chosen later.

If this h satisfies (5.4) then (5.3) and (5.2) imply what we need.

To choose B, p_*, q_* and to prove that h satisfies (5.4) we follow the algorithm below.

Fix an interval I. Consider $f_1 = f \cdot \chi_{2I}, g_1 = g \cdot \chi_{2I}$ and $f_2 = f - f_1, g = g - g_1$. Let us introduce A_I as in (3.1)

$$t \in I, \qquad S^{A_I}(f_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|\nabla P_+ A_I W^{-1/p} f_i\|^2 dA(z) \right)^{1/2}, \ i = 1,2;$$

$$t \in I, \qquad S^{A_I^{-1}}(g_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|\nabla P_+ A_I^{-1} W^{1/p} g_i\|^2 dA(z) \right)^{1/2}, \ i = 1,2.$$

Then exactly as in Section 3 we choose $\alpha > 1$, α close to 1 and

$$\begin{aligned} \|S^{A_I}(f_1)\|_{L^\alpha(I,\frac{dt}{I})} \ &\leq \ C(\alpha,d) \left(\frac{1}{I} \int_{2I} \|f\|^{(p-\delta)\alpha} \right)^{\frac{1}{(p-\delta)\alpha}} \\ &\quad \cdot \sum_{i=1}^{d} \left(\frac{1}{I} \int_I \|W^{-1/p}(t) A_I e_i\|^{(q+\varepsilon)\alpha} dt \right)^{\frac{1}{(q+\varepsilon)\alpha}} . \end{aligned}$$

We can use now the fact that the function $t \to \|U^{1/q}(t)x\| = \|W^{-1/p}(t)x\|$ is uniformly in A_∞ to conclude that we have a uniform inverse Hölder inequality with $\bar{q} > q$. Now choosing ε and α to satisfy $(q+\varepsilon)\alpha < \bar{q}$ we conclude that the last integral is bounded by $\left(\frac{1}{I}\int_I \|W^{-1/p}A_I e_i\|^q dt\right)^{1/q}$. But $W \in A_p$, so A_I^{-1} reduces averages of metric ρ^* given by $W^{-1/p} = U^{1/q}$. In other words

$$\left(\frac{1}{I}\int_I \|W^{-1/p}x\|^q\right)^{1/q} \asymp \|A_I^{-1}x\|. \tag{5.6}$$

Thus finally

$$\|S^{A_I}(f_1)\|_{L^\alpha(I,\frac{dt}{|I|})} \le C'(\alpha,d)\inf_{t\in I}(M\|f\|^{p_*})^{1/p_*}(t)$$

where $p_* = (p-\delta(\varepsilon))\alpha$ is ensured to be in $(1,p)$. Similarly

$$\|S^{A_I^{-1}}(g_1)\|_{L^\alpha(I,\frac{dt}{|I|})} \le C'(\alpha,d)\inf_{t\in I}(M\|g\|^{q_*})^{1/q_*}(t)$$

Now let us estimate $S^{A_I}(f_2)(t), S^{A_I^{-1}}(g_2)(t)$, for $t \in I$.

Let C_I be a center of square Q_I built on I. It is easy to see that for $t \in I$ we have

$$\left(\int_{\Gamma_{t,|I|}} \|(\nabla P_+ A_I W^{-1/p}(t)f_2(t))(z)\|^2\right)^{1/2} \le a\int P_{C_I}(t)\|A_I W^{-1/p}(t)f_2(t)\|dt, \tag{5.7}$$

where P_{C_I} denotes the Poisson kernel. The small δ and let $(p-\delta)^{-1} + (q+\varepsilon)^{-1} = 1$. We can continue the estimate as follows ($k_a^{(r)} \stackrel{def}{=} (1-|a|^2)^{\frac{1}{2}-\frac{1}{r}} \cdot k_a$)

$$\begin{aligned}
\int P_{C_I}\|A_I W^{-1/p} f_2\| &\le \sum_{i=1}^d \int \|W^{-1/p}A_I e_i\|\|f_2\||k_{C_I}|^2(t)\\
&\le C(p,q)\sum_{i=1}^d \int \|W^{-1/p}A_I e_i\|\|f_2\| k_{C_I}^{(q+\varepsilon)} k_{C_I}^{(p-\delta)}\\
&\le C(p,q)\sum_{i=1}^d \left(\int \|W^{-1/p}A_I e_i\|^{q+\varepsilon}|k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon}\right)^{1/q+\varepsilon}\\
&\quad \cdot \left(\int \|f_2\|^{p-\delta}|k_{C_I}^{(p-\delta)}|^{p-\delta}\right)^{\frac{1}{p-\delta}}.
\end{aligned} \tag{5.8}$$

Notice that $|k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon} \le C|k_{C_I}^{(q)}|^q \stackrel{def}{=} k_{I,q}$.

As $t \to \|W^{-1/p}(t)x\|^q$ is uniformly A_q weight (see Lemma 5.3) we can conclude that there exists a small ε such that $t \to \|W^{-1/p}(t)x\|^{q+\varepsilon}$ is uniformly A_q weight by a classical inverse Hölder inequality. Lemma 5.2 claims that if u is a scalar weight in A_q then

$$\int u\, k_{I,q} \le C\frac{1}{I}\int_I u.$$

Combining this knowledge we conclude that

$$\begin{aligned}
&\sum_{i=1}^d \left(\int \|W^{-1/p}A_I e_i\|^{q+\varepsilon}|k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon}\right)^{\frac{1}{q+\varepsilon}} \\
&\qquad \le C\sum_{i=1}^d \left(\frac{1}{I}\int_I \|W^{-1/p}A_I e_i\|^{q+\varepsilon}\right)^{\frac{1}{q+\varepsilon}} \\
&\qquad \le C\sum_{i=1}^d \left(\frac{1}{I}\int_I \|W^{-1/p}A_I e_i\|^{q}\right)^{1/q}.
\end{aligned}$$

The last inequality is another application of inverse Hölder inequality.

Finally, using (5.6) we estimate the last sum by a constant.

Combining this with (5.7) and (5.8) we get ($p_* = p - \delta$)

$$t \in I \Rightarrow S^{A_I}(f_2)(t) \le C(p,q,d)\,(M\|f\|^{p_*})^{\frac{1}{p_*}}(t).$$

Similarly

$$t \in I \Rightarrow S^{A_I^{-1}}(g_2)(t) \le C(p,q,d)(M\|g\|^{q_*})^{1/q_*}(t).$$

Combining with our previous estimates for $S^{A_I}(f_1)$, $S^{A_I^{-1}}(g_1)$ we get

$$\begin{aligned}
\|S^{A_I}(f)\|_{L^\alpha(I,\frac{dt}{|I|})} &\le C(p,q,\alpha,d)(M\|f\|^{p_*})^{1/p_*}(t), \\
\|S^{A_I^{-1}}(g)\|_{L^*(I,\frac{dt}{|I|})} &\le C(p,q,\alpha,d)(M\|g\|^{q_*})^{1/q_*}(t).
\end{aligned}$$

In particular, on a quarter of I

$$
\begin{aligned}
S_I(t) &\stackrel{def}{=} \int_{\Gamma_{t,|I|}} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z) \\
&= \int_{\Gamma_{t,|I|}} |(A_I \nabla P_+ W^{-1/p} f, A_I^{-1} \nabla P_+ W^{1/p} g)| dA(z) \\
&= \int_{\Gamma_{t,|I|}} |(\nabla P_+ A_I W^{-1/p} f, \nabla P_+ A_I^{-1} W^{1/p} g)| dA(z) \\
&\leq S^{A_I}(f)(t) S^{A_I^{-1}}(g)(t) \leq C^2(p,q,\alpha,d) \\
&\quad \cdot (M\|f\|^{p_*})^{1/p_*}(t)(M\|g\|^{q_*})^{1/q_*}(t).
\end{aligned}
$$

For such t, $h(t) \geq |I|$, if we choose in the definition of h to have $B = C^2(p,q,\alpha,d)$ and p_*, q_* as above. Thus (5.4) is satisfied and the Theorem is proved. ■

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