Rigidity of Harmonic Measure

I.Popovici, A. Volberg[1]
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA

Abstract

Let J be the Julia set of a conformal dynamics f. Provided that f is polynomial-like we prove that the harmonic measure on J is mutually absolutely continuous with the measure of maximal entropy if and only if f is conformally equivalent to a polynomial. This is no longer true for generalized polynomial-like maps. But for such dynamics the coincidence of classes of these two measures turns out to be equivalent to the existence of the conformal change of variable which reduces the dynamical system to another one for which harmonic measure equals the measure of maximal entropy.

1 Introduction

The harmonic measure in a dynamical context appeard for the first time in Brolin's paper [Br] where it was established that harmonic measure ω associated with the unbounded Fatou component $A_\infty(f)$ of a polynomial f is balanced, which means that its Jacobian is constant. Later when the ergodic theory of rational maps was developed by Lyubich [Ly1], [Ly2] and independently by Freire, Lopes, Mane [FLM],[M], this balanced measure was interpreted as the unique measure of maximal entropy m .

When we have a more general than polynomial dynamical system the natural question of comparison of these two measures arises. For rational f it was considered by Lopes in [Lo], where it was proved that if $\infty \in \mathbb{C} \setminus J(f)$ is a fixed point of f, then it follows from $m = \omega$ that f is a polynomial. We

[1]Partially supported by NSF Grant DMS 9302728

will consider the local setting of the problem when f is defined only on a neighborhood of an invariant compact set J_f. The question is to characterize the situation when $\omega \approx m$, where "$\approx$" denotes mutual absolute continuity. It certainly happens when f is conformally equivalent to a polynomial. In this paper we discuss the converse problem in the case when f is a *generalized polynomial-like map* (GPL).

Let us pass to precise definitions. Let $U, U_1, U_2, \dots U_k$ be $k+1$ topological discs with real analytic boundaries such that $\bar{U}_i \subset U$, $i = 1, 2, \dots k$; $\bar{U}_i \bigcap \bar{U}_j = \emptyset, i \neq j$. Consider a map $f : \bigcup_{i=1}^k U_i \to U$ which is a branched covering of degree $d_i < \infty$ on each U_i. Then $K_f = \bigcap_{n \geq 0} U^n$, where $U^n = f^{-n}(U)$. We call $J_f = \partial K_f$ the Julia set of f. It is also the boundary of $A_\infty(f) = \bar{\mathbb{C}} \setminus K_f$. The degree of f is $d = d_1 + d_2 + \cdots + d_k$. Let us say that f is polynomial-like (in the sense of Douady and Hubbard [DH]) if $k = 1, d = d_1 \geq 2$.

Saying that two maps f, g are (conformally) conjugate we mean that there is a (conformal) conjugation in some neighborhoods of the Julia sets. Any GPL is quasiconformally conjugate to a polynomial and pulling back the measure of maximal entropy of this polynomial we obtain $m = m_f$, the measure of maximal entropy for f. We call in this paper a GPL g maximal if $\omega_g = m_g$, where ω_g denotes the harmonic measure evaluated at infinity. As infinity has no special role for GPL we introduce the following class of GPL.

We call f *conformally maximal* if there exists another GPL g which is maximal and if there exists a conformal map $h : U_f \to U_g$ which conjugates the dynamics: $f = h^{-1} \circ g \circ h$.

In this paper we prove the following theorems.

Theorem 1 Let f be a GPL. Then the folowing assertions are equivalent:

1. f is conformally maximal
2. $\omega_f \approx m_f$. □

Theorem 2 Let f be a polynomial-like map. The following are equivalent:

1. f is conformally conjugated to a polynomial
2. $\omega_f \approx m_f$. □

To prove these results we will use the technique of [LyV], [BV] and the following result proved by the first author:

Theorem 3 Let f be a GPL. Then there exists a finite measure ν on J_f such that ν is f-invariant and $\nu \approx \omega_f$ $\square$

Theorem 3 was proved independently by A. Zdunik [Z2]. She also derived Theorems 1, 2 from it. We give here our very short proof of Theorem 3.

Theorems 1, 2 were proved by Z. Balogh [B] in the case when J_f is totally disconnected without any assumptions of hyperbolicity. His proof is very different from the one here or from Zdunik's proof (which also differs from our proof). In particular he does not need to construct the invariant harmonic measure (Theorem 3) in [B].

Let us mention that the results of Theorems 1, 2 and 3 were stated as conjectures in [LyV], [BV], [BPV] where particular cases were proved (e.g. semihyperbolic case in [BV]).

2 Existence of invariant harmonic measure

We will use the following result of Y. N. Dowker and A. Calderon which we found in [Fo]. We formulate it in a convenient form.

Theorem A Let μ be a probability measure on a compact set X. Let $T : X \to X$ be a continuous endomorphism such that μ is completely non-singular with respect to T. Then there exists a T-invariant probability measure λ absolutely continuous with respect to μ if and only if the following holds

$$\mu(E) < 1 \Rightarrow \sup_n \mu(f^{-n}E) < 1$$

If μ is ergodic then λ is ergodic. $\square$

To use Theorem A for $X = J_f, \mu = \omega_f$ we are going to estimate $\omega(f^{-n}E)$ via $\omega(E)$.

Fix an arbirary Borel set $E \subset J$, $\omega(E) = 1 - \epsilon < 1$. Let Γ be a smooth curve encircling J and separating it from ∂U and let $\Gamma^n = f^{-n}(\Gamma)$. The main things now are six notations.

Let ω, v denote the harmonic measure of E with respect to $A_\infty(f)$, $U \setminus K_f$ respectively. Let V, W denote the harmonic measures of $J \setminus E$

with respect to $A_\infty, U \setminus K_f$ respectively. For any function ϕ let ϕ^n denote $\phi \circ f^n$ where defined. As usual $\omega_\Omega(S,z)$ denotes the harmonic measure of S evaluated at z with respect to Ω.

First we need a simple lemma. Fix a compact set K in Ω and consider two harmonic measures on $\partial\Omega$ - with respect to Ω and with respect to $\Omega \setminus K$ evaluated at the same point $a \in \Omega \setminus K$.

Lemma 2.1 The two harmonic measures on $\partial\Omega$ are boundedly equivalent. $\square$

Proof: We present the proof in the case when all points of $\partial\Omega$ are regular. Only this case is used in what follows. Let O be a neighborhood of $\partial\Omega$ with smooth boundary and satisfying $O \bigcap (K \bigcup \{a\}) = \emptyset$. Let G, g be Green's functions of $\Omega, \Omega \setminus K$ respectively, with pole at a. Then by Harnack's principle $cG \leq g \leq G$ on the boundary of O. By maximum principle this inequality extends to O (both functions vanish on $\partial\Omega$). Then clearly the functions $g - cG$ and $G - g$ are subharmonic in O and so their Riesz measures are nonnegative. The Riesz measures of G and g being equal to our harmonic measures, we are done.

Proof of Theorem 3: We have to prove that

$$\omega_{A_\infty}(E, \infty) \leq 1 - \epsilon \Rightarrow \omega_{A_\infty}(f^{-n}E, \infty) \leq 1 - \delta.$$

As $\omega(\infty) \leq 1 - \epsilon$, we get $W(\infty) \geq \epsilon$.

Then $W(\xi) \geq c\epsilon$ on Γ and by Lemma 2.1 $V(\xi) \geq \delta$ on Γ. Then $V^n(\xi) \geq \delta$ on Γ_n. But these functions vanish on ∂U^n and so $\omega_{A_\infty}(f^{-n}(J \setminus E), \xi) \geq \delta$ on Γ_n. So $\omega_{A_\infty}(f^{-n}(E), \xi) \leq 1 - \delta$ on Γ_n. As Γ_n separates J from ∞ we obtain $\omega(f^{-n}E) = \omega_{A_\infty}(f^{-n}(E), \infty) \leq 1 - \delta$.

We are going to prove that the just constructed *invariant harmonic measure* ν is boundedly equivalent to ω.

Theorem 4 There exist constants $0 < c_1, c_2 < \infty$ such that

$$c_1 \leq \frac{d\nu}{d\omega} \leq c_2 \quad \square$$

Proof: Let us prove first the right inequality. We wish to repeat the above considerations but it seems hard to get rid of the influence of ∂U. However we are going to prove that $\omega(E) \leq \epsilon \Rightarrow \omega(f^{-n}E) \leq c\epsilon$ for a certain finite c. The construction of invariant measure in [Fo] then gives the right inequality.

Let us fix Γ as above and such that

$$\max_{\xi\in\Gamma}\omega_{U\setminus K_f}(\partial U,\xi)\le\frac{1}{2}.$$

Then if Ω_n is any component of U^n and $\gamma_n = \Gamma^n\bigcap\Omega_n$, we can write $\omega_{U\setminus K_f}(\partial U, f^n(\xi)) = \omega_{\Omega_n\setminus K_f}(\partial\Omega_n,\xi)$ and thus

$$\omega_{\Omega_n\setminus K_f}(\partial\Omega_n,\xi)\le\frac{1}{2},\quad \xi\in\gamma_n \tag{2.1}$$

We start with the chain of implications:

$$\omega(\infty)\le\epsilon\ \Rightarrow\omega(\xi)\le C_\Gamma\epsilon\quad\text{for }\xi\in\Gamma\ \Rightarrow$$

$$\Rightarrow v(\xi)\le C_\Gamma\epsilon\ \text{ for }\xi\in\Gamma\ \Rightarrow v^n(\xi)\le C_\Gamma\epsilon\ \text{ for }\xi\in\gamma_n.$$

Let us compare $u_1(\xi) = v^n(\xi)$, with $u_2(\xi) = \omega_{A_\infty}(f^{-n}E,\xi)$ on γ_n for each component Ω_n of U^n. By Poisson formula in $\Omega_n\setminus K_f$ we have:

$$u_2(\xi) - \int_{\partial\Omega_n} u_2(\eta)d\omega_{\Omega_n\setminus K_f}(\eta,\xi) = u_1(\xi)$$

Let $u_2(\xi_0) = \max_{\gamma_n}\max_{\xi\in\gamma_n}u_2(\xi) = \max_{\xi\in\Gamma^n}u_2(\xi)$.

Then By (2.1) we have $u_2(\xi_0)(1-\frac{1}{2})\le u_1(\xi_0)\le C_\Gamma\epsilon$. But Γ^n separates J_f from ∞ and so

$$\omega_{A_\infty}(f^{-n}E,\infty) = u_2(\infty)\le 2C_\Gamma\epsilon$$

The left inequality can be proved exactly the same way.

3 Automorphic Harmonic Function

Theorem 1 Let f be a GPL. Then the following assertions are equivalent:

1. f is conformally maximal
2. $\omega_f\approx m_f$. $\square$

Proof: We will be using the following notations: if u is a subharmonic function then $\mu_u = \Delta u$ is its Riesz measure. Let G be Green's function of $A_\infty(f)$ with pole at ∞. We know that $\mu_G = \omega$. Let $\Phi = \frac{d\mu_{G\circ f}}{d\mu_G}$, $\phi = \log\Phi$.

Clearly Φ is bounded away from zero and infinity (this is just Harnack's inequality essentially). It will be important for us that Φ is the Jacobian of ω with respect to f. Let $\rho = \frac{d\nu}{d\omega}$, $\gamma = \log \rho$. the measures ν, m are finite and invariant and m is ergodic. So $\nu \approx m$ implies $\nu = m$. We start with the homology equation:

$$\phi - \log d = \gamma \circ f - \gamma, \quad \omega - a.e \text{ on } J \tag{3.1}$$

This is obvious from the computation of the Jacobians of the measures $\nu = m$.

To prove our result it is sufficient (and necessary) to construct *the automorphic harmonic function* τ ([LyV], [BV],[BPV]), i.e. a function satisfying the following:

1) τ is subharmonic in U, $\tau \geq 0$
2) τ vanishes on K_f; $\tau > 0$ on $U \setminus K_f$
3) τ is harmonic in $U \setminus K_f$
4) $\tau(fz) = d\tau(z)$.

To construct such a function we first construct a disc $B = B(x, r)$ centered at the Julia set and a nonnegative subharmonic function u in B such that

$$\frac{d\omega}{d\mu_u} = e^{\gamma} \quad \text{on } B \tag{3.2}$$

Let $F, \omega(F) > 0$ be be a set on which γ is continuous. Let $F_0 \subset F$, $\omega(F_0) > 0$ be such that

$$\lim_{\epsilon \to 0} \frac{\omega(B(y, \epsilon) \bigcap F)}{\omega(B(y, \epsilon))} = 1 \tag{3.3}$$

We use the natural extension $(\tilde{J}, \tilde{f}, \tilde{m})$ of (J, f, m). Here $\tilde{f}$ is just the left shift. We denote by $\pi : \tilde{J} \to J$ the projection onto the "0" coordinate. Then $\tilde{m}(\pi^{-1}(F_0)) > 0$ and by the ergodicity of $\tilde{m}$ one can choose $\tilde{x}$ such that $\tilde{f}^{-n}(\tilde{x}) \in \pi^{-1}(F_0)$ with positive frequency. In particular we have chosen $x \in J_f$ and the sequence of compatible inverse images x_n of x such that $x_n \in F_0$ with positive frequency. But one can do more (see [FLM],[Z1]): we can choose $B = B(x, r)$ such that on $3B$ there are univalent compatible inverse brances F_n such that

$$\text{diam} F_n(2B) \leq e^{-n\delta}$$

$$x_n = F_n(x) \text{ meets } F_0 \text{ with positive frequency} \tag{3.4}$$

Let us consider the family $u_n = d^n G \circ F_n$ in $3B$. Then by (3.1)

$$\frac{d\omega}{d\mu_{u_n}}(y) = e^{\gamma(y)-\gamma(F_n(y))},\ y \in J_f \bigcap 3B.$$

In particular $\| \mu_{u_n} \| \leq C < \infty$ and moreover $c_1\omega \leq \mu_{u_n} \leq c_2\omega$ (see Theorem 4 which gives the boundedness of γ).

Let $\{n_k\}$ be a subsequence such that $\gamma(x_{n_k}) \to c,\ k \to \infty,\ x_{n_k} \in F_0$. Without loss of generality we can think that

$$|\gamma(x_{n_k}) - c| \leq 2^{-k},$$

$$\frac{\omega(F_{n_k}(2B) \setminus F)}{\omega(F_{n_k}(2B))} \leq 2^{-k}$$

The last assertion follows from (3.3), (3.4). Let $E_k = F_{n_k}^{-1}(F \bigcap F_{n_k}(2B))$. Then

$$|\frac{d\omega}{d\mu_{n_k}} - e^{\gamma - c}| \leq C_1 2^{-k} \text{ on } E_k \tag{3.5}$$

$$\mu_{n_k}(2B \setminus E_k) \leq C_2 2^{-k} \tag{3.6}$$

Let K be a relatively closed subset of the disc B. We denote by $S_+(B,K)$ the set of bounded subharmonic functions in B vanishing on K and positive and harmonic in $B \setminus K$. We use two standard lemmas.

Lemma 3.1 Let $\{v_j\}$ be a sequence of uniformly bounded functions from $S_+(B,K)$. Let K be regular for the Dirichlet problem in $\mathbb{C} \setminus K$. Then there exists a subsequence which converges pointwisely to a function from $S_+(B,K)$. $\square$

Lemma 3.2 Let u belong to $S_+(2B,K)$, for some ball $2B$ of diameter less than 1 and having $\operatorname{cap}(B \bigcap K) > 0$. Then

$$\sup_B u \leq C_{B,K} \| \mu_u \| . \ \square$$

So as $\| \mu_{u_{n_k}} \| \leq C$ we conclude that u_{n_k} are uniformly bounded. We may think that the subsequence in Lemma 3.1 is $\{u_{n_k}\}$ itself; put

$u_0 = \lim_{k\to\infty} u_{n_k}$ in B. The convergence is pointwise bounded and so $\mu_{u_{n_k}} \to \mu_{u_0}$ weakly. But (3.6), (3.7) show that $\mu_{u_{n_k}} \to e^{-\gamma+c} d\omega$ weakly. Thus $d\mu_{u_0} = e^{-\gamma+c} d\omega$ and $u_0 e^c$ satisfies (3.2).

Now the construction of τ follows word by word the construction in [LyV],[BV]. We sketch it here for the sake of completness. We consuder B_θ a component of $f^{-n}B$ and define τ on it as follows:

$$\tau(z) \stackrel{def}{=} \frac{1}{d^n} u(f^n z)$$

In [LyV] or [BV] it is shown that τ (or its symmetrization) does not depend on θ or n, that is if $B_{\theta_1} \bigcap B_{\theta_2} \neq \emptyset$ then τ (or its symmetrization) is the same on this intersection. This follows quite easily from (3.2). Now τ is defined on the set $O = \bigcup f^{-n} B$. Using (2.2) it is easy to extend it to U. Theorem 1 is therefore proved.

Proof of Theorem 2: The circular model of Douady and Hubbard associates to a polynomial-like map f an expanding real analytic endomorphism F of degree d of the unit circle $\mathbb{T}$. If f is a polynomial, then $F : z \to z^d$. Polynomial-like maps are called *externally equivalent* if the corresponding endomorphisms are analytically conjugate.

Theorem B ([DH]) Let f be a polynomial-like map of degree d. Then f is conformally equivalent to a polynomial if and only if it is externally equivalent to $z \to z^d$. $\Box$

An automorphic harmonic function τ built in Section 3 gives this external equivalency easily. This is done in [LyV], but we will sketch the proof here. Let F be a Douady-Hubbard external map for f, then $F : A_1 \to A_2$ is an analytic covering of degree d between annuli $A_1, A_2, \bar{A}_1 \subset A_2, \mathbb{T} \subset \partial A_1 \bigcap \partial A_2$. Given the f-automorphic harmonic function τ we can pull it on A_1 thus constructing an F-automorphic harmonic functionn T on A_1; $T(Fz) = dT(z)$. Then T vanishes on the unit circle. We can extend T through the circle by reflection to obtain a subharmonic function with $\Delta T = \rho d\sigma$, where σ denotes the Lebesque measure on the circle and ρ is real analytic. But the automorphic equation gives that the Riesz measure ΔT of T is a maximal measure for F. Thus m_F is equivalent with the Lebesque measure and a theorem of Shub and Sullivan [SS] finishes the proof by showing that F is analytically equivalent to $z \to z^d$.

4 Questions

We finish by formulating two questions:

1. Can one use the technique of [MR] to prove the theorems above ?
2. Given a polynomial-like map f and its external map $F : \mathbb{T} \to \mathbb{T}$ is it true that the harmonic measure on J_f is transfered to a measure absolutely continuous with respect to Lebesque measure on the circle?

Positive answers would give an easy proof of Theorem 2. But it will not be possible to use it for Theorem 1 without expanding the Douady-Hubbard external model to GPL.

5 Non-linearizability of harmonic measure

The method used for the proof of Theorem 1 gives that the logarithm of the Jacobian of harmonic measure ϕ can satisfy $\phi - \psi = \gamma \circ f - \gamma$ with continuous ψ having only finitely many values only if ψ is actually a constant function, equal to $\log d$, and f is conformally maximal.

References

[B] Z.Balogh, *Rigidity of harmonic measure on totally disconnected fractals*, Preprint, Michigan State University, April 1995.

[BPV] Z.Balogh, I. Popovici, A. Volberg, *Conformally maximal polynimial-like dynamics and invariant harmonic measure*, Preprint, Uppsala Univ., U.U.D.M. Report 1994:32.

[Br] H.Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. 6, (1965), 103-145.

[BV] Z. Balogh, A. Volberg, *Principe de Harnack à la frontière pour les repulseurs holomorphes non-recurrents*, C.R. Acad. Sci. Paris, t. 319, (1994), 351-35.

[DH] A. Douady, F. Hubbard, *On the dynamics of polynomial like mappings*, Ann. Sci. Ec. Norm. Sup. 1985, v. 18, No. 2, pp. 287-345.

[Fo] S.R.Foguel, *The ergodic theory of Markov processes*, Math. Studies, v. 21, New York: Van Nostrand, 1969.

[FLM] A. Freire, A. Lopes, R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Bras. Mat. (1983), v. 14, No. 1, 45-62.

[Lo] A. Lopes, *Equilibrium measure for ratational functions,* Erg. Th. & Dyn. Syst., 1986, 6, 393-399.

[Ly1] M. Lyubich, *Entropy of the analytic endomorphisms of the Riemann sphere*, Funk. An. and Appl. 1981, 15:4, 83-84.

[Ly2] M. Yu. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory and Dynamical Systems 3, (1983), 351-386.

[LV] M. Yu. Lyubich, A. Volberg, *A comparisson of harmonic and maximal measures on Cantor repellers*, The Journal of Fourier Analysis and Applications, (1995).

[M] R Mañe, *On the uniqness of the maximizing measure for rational maps*, Bol. Soc. Brasil. Mat. 14, (1983), 27-43.

[MR] R. Mañé, L.F. da Rocha,*Julia sets are uniformly perfect*, Proc. AMS, 1992, v. 116, 251-257.

[SS] M. Shub, D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergodic Theory and Dynamical Systems, 1985, v. 5, 285-289.

[Z1] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Mat. 99, (1990), 627-649.

[Z2] A. Zdunik, *Invariant Measure in the class of harmonic measure for polynomial-like mappings*, Preprint Inst. Math. Polish Ac. of Sci., 1995:538, 1-21.