Counterexamples to infinite dimensional Carleson imbedding theorem[*]

F. Nazarov, S. Treil[*], A. Volberg[*]

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First draft. A very preliminary version

1 Description of result.

Let be the unit circle and $\{I\}_{I\in\mathcal{D}}$ be the collection of its dyadic arcs. Let m denote Lebesgue measure on and let μ be a positive Borel measure in the unit disc . The classical Carleson imbedding theorem [G] implies the following result in dyadic setting.

Consider the squares Q_I built on I in and let TQ_I denote the top half of Q_I. Consider the imbedding operator.

$$f \in L^2(), \quad f \xrightarrow{T} \sum_{I\in\mathcal{D}} \left(\frac{1}{I} \int_I f\, dm \right) \chi_{TQ_I}(z)$$

Theorem A. *The operator T is bounded from $L^2()$ to $L^2(, \mu)$ if and only if*

$$\mu(Q_J) \le Cm(J) = C|J|. \tag{1.1}$$

Corollary B. *The imbedding operator is bounded from $L^2()$ to $L^2(\mu)$ if and only if it is uniformly bounded on all test functions $f = \chi_J,\ J \in \mathcal{D}$.*

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In fact, obviously $|J| = \|\chi_J\|^2_{L^2()}$; $\mu(Q_J) \le \|T\chi_J\|^2_{L^2(\mu)}$. There are more difficult results where one considers the imbedding with the weighted input space. In other words one replaces $L^2()$ by $L^2(, w)$, $w \ge 0$, $w \in L^1$.

Theorem C. *The operator T is bounded from $L^2(, w)$ to $L^2(, d\mu)$ if and only if it is uniformly bounded on all test functions $f = \chi_J w^{-1}$, $J \in \mathcal{D}$.*

This is the result of E. Sawyer [S1], [S2]. Different proofs can be found in [NT], [TV1].

Recently some progress has been made in understanding vector singular integral operators with matrix weights. In particular, if W is a positive definite $d\times d$ matrix-function on (a matrix weight) one can characterize those W which make Calderón-Zygmund operators bounded in $L^p(^d, W)$ $(1 < p < \infty)$ [TV2], [NT], [V]. One of the main ideas in [TV2] is to reduce the singular integral estimate to a weighted imbedding theorem of Carleson-Sawyer type. But quite naturally the measures $W dm$ and $d\mu$ are matrix valued measures now.

So let $W(t)$ be a positive definite $d \times d$ matrix for a.e. $t \in$. Let μ be a matrix with entries which are measures, and $\mu(E)$ be a positive definite $d\times d$ matrix for any Borel measurable E in .

Theorem D. *The imbedding operator T which acts coordinatewise on $L^2(^d; W\, dm)$ is bounded into $L^2(^d; , d\mu)$ if and only if it is uniformly bounded on all test vector functions $f = W^{-1}e\,\chi_J$, $J \in \mathcal{D}$, $e \in^d$.*

This kind of result played a key role in [TV2]. It was not stated explicitely there. We are going to give a proof elsewhere.

Now we are interested in infinite dimensional $(d = \infty)$ version of the imbedding theorem. The reason is clear: if one wants to proceed as in [TV2], [NT], [V] to obtain the criterion for the boundedness of Calderón-Zygmund operators with *operator*-valued weights W ($W = W(t)$, $t \in$, $W(t) : H \to H$ is a positive operator in an infinitely dimensional Hilbert space H) one wishes to have the imbedding theorem for Hilbert space valued functions.

So a natural question arises.

Question. Is that so that coordinatewise imbedding operator T, $Tf = \sum_{I\in\mathcal{D}} \left(\frac{1}{I}\int_I f\, dm\right) \chi_{TQ_I}(z)$, is bounded from $L^2(H; dm)$ to $L^2(H; , d\mu)$ if and only if it is uniformly bounded on all test functions $f = e\chi_J$, $J \in \mathcal{D}$, $e \in H$?

If the answer would be affirmative one would ask this question for weighted input space $L^2(H; W\,dm)$ as well.

But the goal of this work is to show that the answer to the question is *NEGATIVE*. The counterexamples are quite tricky for a good reason – the growth of $\|T\|$ with respect to the dimension seems to be very slow.

2 Counterexample to the infinite dimensional Carleson imbedding theorem

It is no wonder that this counterexample appeared in MSU. It is based on the "equality"

$$\text{``}(a+b)^2 = a^2 + b^2\text{''},$$

which is engraved deeply in the hearts of our students. In what follows this "equality" or rather its correct counterpart

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$$

will play an important part implying lots of cancellations. Again we are on the unit interval and let $\mathcal{D}_{||}$ denote its dyadic subintervals of the length 2^{-k}. The operator valued measure μ in will be searched in the form

$$d\mu = \sum_{k=0}^{n} \sum_{I \in \mathcal{D}_k} (\cdot, {}_I)_I |I| \{c_I\},$$

where c_I is the center of TQ_I, $\{c_I\}$ is the point measure in c_I and ${}_I$ are certain unit vectors in H to be found to have a good imbedding on test functions and a bad imbedding on a certin $L^2(H; dm)$-function. To construct ${}_I$ let us first choose a sequence $\{a_j\}_{j=0}^{\infty}$ of positive numbers to be fixed later but such that

$$\sum a_j^2 = 1. \tag{2.1}$$

Now let us consider the nest of dyadic intervals $\{I, I, \dots, {}^n I\}$, where $I \in \mathcal{D}_n$ and maps any interval to his "father". Every ${}_{kI}$ will have the following form

$${}_{kI} = \sum_{j=0}^{\infty} \pm a_j e_{n-k-j}, \tag{2.2}$$

where $\{e_m\}_{m=-\infty}^{\infty}$ is a fixed orthonormal basis in H. We did not fix the signs $\pm$ in the definition of $_{^kI}$. Their choice is dependent on $I \in \mathcal{D}_n$ and $k \in [0, n]$ and the next section is devoted to this choice.

2.1 Sign choice pattern

Let G be a dyadic graph in $Q_{[0,1]}$. So its vertices are dyadic subintervals J of $[0,1]$ and edges connect fathers with sons. Now if the edge from ${}^{k+1}I$ to ${}^{k}I$ goes in the left direction we choose to put "+" sign if front of $a_0 e_{n-k}$ in expansion (2.2) of $_{^kI}$, "+" in front of $a_1 e_{n-k}$ in expansion (2.2) of $_{^{k-1}I}, \dots$, "+" in front of $a_k e_{n-k}$ in expansion (2.2) of $_{I}$. If the edge from ${}^{k+1}I$ to ${}^{k}I$ goes to the right we put the sign "—" in front of the same terms in expansions (2.2) of the same vectors. The sign pattern is illustrated by the following picture on eight first intervals $I_1, I_2, I_3, \dots, I_8$ of $\mathcal{D}_n$ from the left end of $[0,1]$:

$$
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_2 & a_1 & a_0 & a_2 & a_1 & a_0 & a_2 & a_1 & a_0 & a_2 & a_1 & a_0 \\
a_1 & a_0 & & a_1 & a_0 & & -a_1 & -a_0 & & -a_1 & -a_0 & \\
a_0 & & & -a_0 & & & a_0 & & & -a_0 & & \\
 & I_1 & & I_2 & & & I_3 & & & I_4 & &
\end{array}
$$

$$
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-a_2 & -a_1 & -a_0 & -a_2 & -a_1 & -a_0 & -a_2 & -a_1 & -a_0 & -a_2 & -a_1 & -a_0 \\
a_1 & a_0 & & a_1 & a_0 & & -a_1 & -a_0 & & -a_1 & -a_0 & \\
a_0 & & & -a_0 & & & a_0 & & & -a_0 & & \\
I_5 & & & I_6 & & & I_7 & & & I_8 & &
\end{array}
$$

All other "a's" on this diagrams for $I_+, \dots, I_8$ are with sign "+'.

To restore $_{^kI}$, $k = 0, \dots, n$, using these diagrams one does the following: take the first column over I_i ($I_i \in \mathcal{D}_n$) and consider its entries as coefficients for expansion in $e_n, e_{n-1}, \dots$ with the lowest entry being the coefficient in front of e_n, the next lowest entry being the coefficient in front of e_{n-1}, etc... The second column represents similarly the expansion in $e_{n-1}, e_{n-2}, \dots$ And so on. Then the first column is $_{I_i}$, the second one is $_{I_i}$, etc...

2.2 Two main theorems

Theorem 2.1. *If $\sum_{j=1}^{\infty} j\, a_j^2 < \infty$ then the operator measures*

$$d\mu = \sum_{k=0}^{n} \sum_{I \in \mathcal{D}_k} (\cdot,_I)_I |I| \{c_I\} \tag{2.3}$$

are uniformly Carleson measures with respect to n in the sense that there exists a constant C, $C < \infty$, independent of n, such that

$$\forall\, n \forall\, e \in H \,\forall\, J \in \mathcal{D} \sum_{I \subseteq J} |(e,_I)|^2 |I| \leq C|J| \|e\|^2. \tag{2.4}$$

Let P_k denote the orthogonal projection in H onto $V_k \stackrel{def}{=}$ span $(e_k, e_{k-1}, \ldots)$.

Lemma 2.2 *For any $I \in \mathcal{D}_n$*

$$\sum_{k=0}^{n-\ell} \|P_{\ell^k I}\|^2 \leq C < \infty. \tag{2.5}$$

Proof. It is clear that $\|P_{\ell I}\|^2 = a_{n-\ell}^2 + a_{n-\ell+1}^2 + \cdots$; $\|P_{\ell I}\|^2 = a_{n-\ell-1}^2 + a_{n-\ell}^2 + \cdots$; $\cdots$; $\|P_{\ell^{n-\ell} I}\|^2 = a_0^2 + a_1^2 + \cdots$ and (2.5) follows from our assumption $\sum_{j \geq 1} j\, a_j^2 < \infty$.

Here is a trivial observation which will be of a special importance to us.

Lemma 2.3. *For any e, $e_0 \in H$, $v \in H$, $a \in$,*

$$|(e, v + ae_0)|^2 + |(e, v - ae_0)|^2 = 2|(e,v)|^2 + 2|(e,e_0)|^2 a^2.$$

Proof of Theorem 2.1. Fix $e \in H$, $\|e\| = 1$ and $J \in \bigcup_{\ell=0}^{n} \mathcal{D}_\ell$, $J \subseteq [0,1]$. Let $|J| = 2^{-\ell}$.

Let us first estimate the lower layer in (2.4) that is the sum $\sum_{I \in \mathcal{D}_n, I \subset J} |(e,_I)|^2 |I|$. We write $_{I_-} = P_{n-1 I_-} + (n,_{I_-}) a_0 e_n$, $_{I_+} = P_{n-1 I_+} + (n,_{I_+}) a_0 e_n$ for any interval $I \in \mathcal{D}_{n-1}$, where I_-, I_+ are left and right halves of I and where $(n,_I)$ denote the sign in our pattern. The main thing to notice is that the pattern satisfies

$$\left.\begin{aligned} P_{n-1I_-} &= P_{n-1I_+} \\ (n,_{I_-}) &= -(n,_{I_+}) \end{aligned}\right\} \tag{2.6}$$

Relationships (2.6) are the general rules in our pattern: let J', J'' be intervals in $\mathcal{D}_{n-k}$ having the same father J in $\mathcal{D}_{n-k-1}$, then

$$P_{n-k-1J'} = P_{n-k-1J''} \stackrel{def}{=} v_{n,k,0,J}, \tag{2.7}$$

$$({}_{J'} - P_{n-k-1J'}) = -({}_{J''} - P_{n-k-1J''}). \tag{2.8}$$

As

$$_{J'} = P_{n-k-1J'} + (n-k, P_{J'})a_0 e_{n-k},$$

$$_{J''} = P_{n-k-1J''} + (n-k,_{J''})a_0 e_{n-k},$$

we can rewrite (2.7) as

$$(n-k,_{J'}) = -(n-k,_{J''}). \tag{2.9}$$

In what follows we will need a further observation on our sign pattern. Relationships (2.6)-(2.9) are particular cases of the following.

Let J', J'' be two intervals in $\mathcal{D}_{n-k}$, and ${}^\ell J',{}^\ell J''$ have the same father J in $\mathcal{D}_{n-k-\ell-1}$. Then

$$P_{n-k-\ell-1J'} = P_{n-k-\ell-1J''} \stackrel{def}{=} v_{n,k,\ell,J}, \tag{2.10}$$

$$(P_{n-k-\ell} - P_{n-k-\ell-1})_{J'} = -(P_{n-k-\ell} - P_{n-k-\ell-1})_{J''}. \tag{2.11}$$

The last equality means that

$$(n-k-\ell,_{J'}) = -(n-k-\ell,_{J''}). \tag{2.12}$$

These combinatorial properties will soon imply an immense cancellations in sums of type (2.4).

Coming back to (2.4) let us write the sum uniting pairs of sons of the same father. And then let us start to apply Lemma 2.2 repeatedly using (2.7) and (2.9) with $k = 0$.

We certainly could organize the computation below in a different way. It would be sufficient to notice that for $I \in \mathcal{D}_k$, $k = 0, 1, ..., n$ one can write ${}_I$ as

$$
{}_I(t) = \sum_{j=0}^{\infty} r_{n-k-j}(t) a_j e_{n-k-j}, \; t \in I
$$

where $r_j(t)$ stands for the Rademacher function on $\mathbb{R}$. Then

$$
\sum_{I \in \mathcal{D}_n} |(e, {}_I)|^2 \cdot |I| = \int_0^1 |\sum_{j=0}^{\infty} r_{n-j}(t) a_j (e, e_{n-j})|^2 \, dt,
$$

and the orthogonality of Rademacher functions $r_n, r_{n-1}, ..., r_0$ on the unit interval incurs lots of cancellations. More naively but doing exactly the same thing, let us just compute

$$2^{-n}\sum_{I\in\mathcal{D}_{n-1}}(|(e,P_{n-1I_-}+a_0e_n)|^2+|(e,P_{n-1I_+}-a_0e_n)|^2)$$

$$=2^{-n}\sum_{I\in\mathcal{D}_{n-1}}(2|(e,v_{n,0,0,I})|^2+2a_0^2|(e,e_n)|^2)$$

$$=2^{-(n-1)}\#\{I\in\mathcal{D}_{n-1}:I\subset J\}a_0^2|(e,e_n)|^2$$

$$+2^{-(n-1)}\sum_{I\in\mathcal{D}_{n-2}}(|(e,P_{n-2}v_{n,0,0,I_-}+a_1e_{n-1})|^2$$

$$+|(e,P_{n-2}v_{n,0,0,I_+}-a_1e_{n-1})|^2)$$

$$=2^{-\ell}a_0^2|(e,e_n)|^2+2^{-(n-1)}\sum_{I\in\mathcal{D}_{n-2}}(|(e,v_{n,0,1,I}+a_1e_{n-1})|^2$$

$$+|(e,v_{n,0,1,I}-a_1e_{n-1})|^2)$$

$$=2^{-\ell}a_0^2|(e,e_n)|^2+2^{-(n-2)}\#\{I\in\mathcal{D}_{n-2}:I\subset J\}a_1^2|(e,e_{n-1})|^2$$

$$+2^{-(n-2)}\sum_{I\in\mathcal{D}_{n-2}}|(e,v_{n,0,1,I})|^2=2^{-\ell}a_0^2|(e,e_n)|^2$$

$$+2^{-\ell}a_1^2|(e,e_{n-1})|^2+2^{-(n-2)}\sum_{I\in\mathcal{D}_{n-3}}(|(e,P_{n-3}v_{n,0,1,I_-}+a_2e_{n-2})|^2$$

$$+|(e,P_{n-3}v_{n,0,1,I_+}-a_2e_{n-2})|^2)=2^{-\ell}a_0^2|(e,e_n)|^2+2^{-\ell}a_1^2|(e,e_{n-1})|^2$$

$$+2^{-(n-2)}\sum_{I\in\mathcal{D}_{n-3}}(|(e,v_{n,0,2,I}+a_2e_{n-2})|^2+|(e,v_{n,0,2,I}-a_2e_{n-2})|^2)$$

$$=\cdots=2^{-\ell}a_0^2|(e,e_n)|^2+2^{-\ell}a_1^2|(e,e_{n-1})|^2+\cdots+2^{-(n-s)}\#$$

$$\{I\in\mathcal{D}_{n-s}:I\subset J\}a_{s-1}^2|(e,e_{n-s+1})|^2+\cdots+2^{-\ell}|(e,v_{n,0,n-\ell-1,[0,1]})|^2$$

$$\le 2^{-\ell}\sum_{i=1}^{n-\ell}a_{i-1}^2|(e,e_{n-i+1})|^2+2^{-\ell}\|P_{\ell I}\|^2,$$

for arbitrary $I\subset J,\ I\in\mathcal{D}_n$.

This is the estimate of the first layer in (2.4). (First means lowest.) The estimate of the second layer is absolutely similar. We again use Lemma 2.2

repeatedly. Only n is replaced by $n-1$. And in (2.7), (2.9) k now equals 1. So the second layer has the following estimate

$$\sum_{I\in\mathcal{D}_{n-1}} |(e,_I)|^2|I| \le 2^{-\ell}\sum_{i=1}^{n-1-\ell} a_{i-1}^2|(e,e_{n-1-i+1})|^2 + 2^{-\ell}\|P_{\ell I}\|^2,$$

for any $I\subset J,\ I\in\mathcal{D}_{n-1}$.

The same type of estimate holds for the third layer, etc... till the $(n-\ell)$-th layer. (There are no more layers in Q_J, $J\in\mathcal{D}_\ell$.) Summing up over all these layers, that is summing up the following sums (in which $I\subset J$ for a fixed $J\in\mathcal{D}_\ell$):

$$\sum_{I\in\mathcal{D}_n}|(e,_I)|^2|I|,\dots,\ \sum_{I\in\mathcal{D}_{\ell+1}}|(e,_I)|^2|I|,\ |(e,_J)|^2|J|$$

we get that the left part of (2.4) can be estimated as follows ($|J|=2^{-\ell}$ as you remember):

$$\sum_{I\subseteq J}|(e,_I)|^2|I| \le 2^{-\ell}\sum_{r=0}^{n-\ell-1}\sum_{i=1}^{n-\ell-r} a_{i-1}^2|(e,e_{n-r-i+1})|^2$$

$$+2^{-\ell}\sum_{k=0}^{n-\ell}\|P_{\ell^k I}\|^2 \stackrel{def}{=} 2^{-\ell}\sum_1 + 2^{-\ell}\sum_2, \tag{2.13}$$

where I is any interval in $\mathcal{D}_n,\ I\subset J$.

Now let us use the fact that $\{e_m\}_{m=-\infty}^{\infty}$ form an orthonormal basis in H. Then we can estimate the first sum in the right part of (2.13) as follows (see also (2.1)):

$$\sum_1 \le \sum_{k=1}^{\infty} a_k^2\sum_m|(e,e_m)|^2 \le 1.$$

On the other hand Lemma 2.2 estimates the second sum in the right part of (2.13):

$$\sum_2 \le \sum_{k=0}^{\infty}(k+1)a_k^2 \le C < \infty.$$

And so estimate (2.4) follows:

$$\sum_{I\subseteq J}|(e,_I)|^2|I| \le |J|\Big(\sum_{k\ge 0}a_k^2 + \sum_{k\ge 0}(k+1)a_k^2\Big) \le (1+C)|J|. \tag{2.14}$$

Theorem 2.1 is proved.

Theorem 2.4. *Suppose the operator measures*

$$d\mu_n = \sum_{k=0}^{n} \sum_{I\in\mathcal{D}_k} (\cdot, {}_I)_I |I| \{c_I\}$$

are built using the same system $\{_I\}_{I\in\mathcal{D}}$ *as above. If* $\sum_{j=0}^{\infty} a_j = \infty$ *then the embedding operator*

$$f \in L^2(H;), f \overset{T}{\mapsto} \sum (\frac{1}{I} \int_I f\, dm) \chi_{TQ_I}(z)$$

has the norm growing to infinity with n. *More precisely*

$$\|T\|_{L^2(H;)\to L^2(H;,\mu_n)} \geq \frac{1}{\sqrt{n+1}} \sqrt{a_0^2 + (a_0+a_1)^2 + \cdots + (a_0+a_1+\cdots+a_n)^2}. \quad (2.15)$$

Proof. We are going to build an H-valued function $E(t)$, $t \in$, such that $\int \|E(t)\|_H^2 = n+1$ and $\int (d\mu_n TE, TE)_H = \sum_{I\in\mathcal{D}} |(\langle E\rangle_{I}, {}_I)_H|^2 |I| \geq a_0^2 + (a_0+a_1)^2 + \cdots + (a_0+a_1+\cdots+a_n)^2$. Here $\langle E\rangle_I \in H$ naturally denotes the average $\frac{1}{I}\int_I E(t)dm(t)$ of E over I. As $\int \|E(t)\|_H^2 dm(t) = \|E\|^2_{L^2(H;)}$ and $\int (d\mu_n TE, TE) = \|TE\|^2_{L^2(H;,\mu_n)}$, estimate (2.15) would follow.

We are going to choose E in such a way that it will be constant on each $I \in \mathcal{D}_n$ (step function) and in such a way that $\frac{1}{\sqrt{n+1}}E$ is "unimodular" in the sense that

$$\|E(t)\|_H^2 \equiv n+1. \quad (2.16)$$

To understand better the choice of $E(t) \equiv \langle E\rangle_I$, $I \in \mathcal{D}_n$ $t \in I$, the reader should look again at **Section 2.1: Sign choice pattern.**

$$t \in I,\ I \in \mathcal{D}_n,\ E(t) \equiv \sum_{k=0}^{n} \frac{({}_I, e_k)}{|({}_I, e_k)|} e_k. \quad (2.17)$$

In other words $E(t) = \sum_{k=0}^{n} \pm e_k$ and the choice of signs for all $t \in I$, for a given $I \in \mathcal{D}_n$ follows the sign pattern of ${}_I$. Thus for example, on the first eight intervals from the left $I \in \mathcal{D}_n$,

$$
\begin{aligned}
E &\equiv e_n + e_{n-1} + \cdots + e_0 \text{ on the first } I,\\
E &\equiv -e_n + e_{n-1} + \cdots + e_0; \text{ on the second } I,\\
E &\equiv e_n - e_{n-1} + e_{n-2} + \cdots + e_0; \text{ on the third } I,\\
E &\equiv -e_n - e_{n-1} + e_{n-2} + \cdots + e_0; \text{ on the fourth } I,\\
E &\equiv e_n + e_{n-1} - e_{n-2} + e_{n-3} + \cdots + e_0; \text{ on the fifth } I,\\
E &\equiv -e_n + e_{n-1} - e_{n-2} + e_{n-3} + \cdots + e_0; \text{ on the sixth } I,\\
E &\equiv e_n - e_{n-1} - e_{n-2} + e_{n-3} + \cdots + e_0; \text{ on the seventh } I,\\
E &\equiv -e_n - e_{n-1} - e_{n-2} + e_{n-3} + \cdots + e_0; \text{ on the eighth } I.
\end{aligned}
$$

Repeating the same thing again one illustrates the expansion of $E(t)$ in $e_n, e_{n-1}, \ldots, e_0$ by the diagram of expansion coefficients on the first eight dyadic intervals from $\mathcal{D}_n$:

$$
\begin{array}{cccccccc}
 & & & & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8
\end{array}
$$

To estimate $\sum_{J\in\mathcal{D}} |(\langle E\rangle_J, {}_J)|^2|J|$ from below let us consider the following S-function:

$$
S^2(E)(t) \stackrel{def}{=} \sum_{J:t\in J} |(\langle E\rangle_J, {}_J)|^2. \tag{2.18}
$$

Actually it is identically constant. In fact, the pattern of signs of $\langle E\rangle_J$ is exactly the same as for ${}_J$, for any $J \in \mathcal{D}$. For $J \in \mathcal{D}_n$ this claim is just the definition of E (see (2.17)). For other J's it follows easily from examining the sign pattern. In particular, for any $J \in \mathcal{D}_\ell$,

$$
(\langle E\rangle_J, {}_J) = a_0 + a_1 + \cdots + a_\ell. \tag{2.19}
$$

Inserting (2.19) into (2.18) we get

$$S^2(E)(t) \equiv (a_0 + a_1 + \cdots + a_n)^2 + (a_0 + a_1 + \cdots + a_{n-1})^2 + \cdots + a_0^2.$$

On the other hand, we have the following standard formula

$$\int S^2(E)(t)dm(t) = \sum_{J\in\mathcal{D}} |(\langle E\rangle_{J},_{J})|^2|J|$$

which is just the change of order of summation and integration. Thus

$$\sum_{J\in\mathcal{D}} |(\langle E\rangle_{J},_{J})|^2|J| \geq \sum_{\ell=0}^{n}\left(\sum_{k=0}^{\ell} a_k\right)^2 \tag{2.20}$$

which finishes the construction of $E(t)$ and the proof of Theorem 2.2.

2.3 The estimate from below with respect to $\dim H$

One notices that actually we worked in $H_n = \mathrm{span}(e_n, \ldots, e_0)$ of dimension $n+1$. If we are interested in the estimate of

$$\stackrel{def}{=} \sup_{\mu}\left[\|T\|_{L^2(H;)\to L^2(H;,\mu)} / \sup_{\substack{J\in\mathcal{D},\\ e\in H,\|e\|\leq 1}} \{(\mu(Q_J)e,e)/|J|\}^{1/2}\right]$$

from below with respect to the $\dim H \uparrow +\infty$ we can play with the sequence $a_j > 0,\ j = 0, 1, \ldots,$

1) One can start with $a_j = \dfrac{1}{(j+1)\log(j+1)}$ which satisfies

$$\sum j\, a_j^2 < \infty, \tag{2.21}$$

$$\sum a_j = \infty. \tag{2.22}$$

Then estimate (2.15) and Theorem 2.1 give

$$\geq c\sqrt{\log\log(n+1)}.$$