On harmonic measure on the Julia set of Blaschke products with one petal

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1 Introduction

In this paper we are dealing with harmonic measure on the Julia sets of parabolic rational functions. Let $\Omega = \bar{\mathbb{C}} \setminus K$ be an open set on the Riemann sphere $\bar{\mathbb{C}}$ and let $\omega_\Omega(E, z)$ be the harmonic measure of $E \subset K$ with respect to Ω, evaluated at $z \in \Omega$. It can be defined as the representing measure of $\bar{u}(z)$ where $\bar{u}(z)$ is the

Perron solution of the Dirichlet problem with continuous boundary values u. In other words the following formula defines the harmonic measure

$$\bar{u}(z) = \int u d\omega_\Omega(\cdot, z)$$

The compact set K must be of positive capacity but not necessarily with complement regular for the Dirichlet problem.

If z_1 and z_2 belong to the same component of Ω then the corresponding measures are boundedly equivalent. Accordingly if Ω is connected, then the possibility $\omega_\Omega(E, z) = 1$ is independent of z, and a Borel set E with this property is said to be a Borel support of harmonic measure. During last several years there was a considerable interest in the metric properties of such sets. In particular, the estimations (and even calculations) of

$$Hdim(\omega) \stackrel{\text{def}}{=} \inf\{Hdim(E) : E \text{ is a Borel support of harmonic measure } \omega\}$$

have been done. Here the symbol Hdim stands for the Hausdorff dimension. As a result of this attention and especially due to works of Carleson [Ca1]-[Ca3], Makarov [Ma1] , Jones, Wolff [JW1], [Jw2],Wolff [W], and Bourgain [B] the structure of harmonic measure of general plane sets become much more comprehensible. The deep analogy between the behavior of sums of (almost) independent random variables and the behavior of the Green function of the domain plays a crucial part in this subject. We send the reader to [Ma2] to learn this. This analogy becomes still more conspicuous if a domain for which the harmonic measure is investigated has regular self-similar structure. As Carleson showed in [Ca3] the methods of ergodic theory turn out to be relevant in this case. This approach was used also in [P], [Z], [MV], [Vo1], [Vo2]. In this paper we use extensively the ideas of these works. But the parabolic singularity forces us to use a new technical tool, that is the Schweiger formalism [Sch] whose power was demonstrated in [DU1]-[DU3]. In some sense we continue here what was undertaken in [Ca3], [MV], [Vo2] borrowing our general approach from [DU3].

It is necessary to point out that harmonic measure in dynamical context appeared for the first time in Brolin's paper [Br], where it was established that backward orbits of a polynomial f are equidistributed with respect to harmonic measure ω of the unbounded component of the Julia set $J(f)$. Later this result was interpreted as the coincidence of ω and the unique measure of maximal entropy of f ("maximal measure") [Ly], [Mañ].

Julia set of a polynomial f can be a really complicated object. But now we would like to note that the result of Brolin and Manning's formula [Man] gives the following estimate of the Hausdorff dimension of harmonic measure on $J(f)$;

$$Hdim(\omega) = \frac{\log d}{\int_{J(f)} \log |f'| d\omega} \leq 1$$

In fact $\int_{J(f)} \log |f'| d\omega = \log d + \sum_i G(c_i)$, where G is the Green function of $\bar{\mathbb{C}} \setminus J(f)$ and c_i are critical points of f escaping to infinity (that is lying in an unbounded component of the complement of $J(f)$). This solves the conjecture of Oksendal [O] for compacts which are Julia sets of polynomials. The general conjecture was solved in [JW1], where the estimate

$$Hdim(\omega) \leq 1$$

has been proved for *any* compact set K. Note that for $J(f)$ with polynomial f such that there exists at least one escaping critical point one can see exactly as above that

$$Hdim(\omega) < 1$$

The existence of escaping critical point means precisely that $J(f)$ is disconnected. Certainly the situations with connected set are covered by the famous result of Makarov which deals with arbitrary *continuum* K:

$$Hdim(\omega) = 1$$

Coming back to the case when K is a limit set of a holomorphic dynamical system f one can suggest two conjectures:

$$Hdim(\omega) < 1 \tag{1.1}$$

for harmonic measure on $J(f)$ unless $J(f)$ is connected. This is not true for rational functions. However recently Zdunik proved this conjecture for so-called generalized polynomial-like dynamics (GPL) f. See [LV] for the definition of GPL.

Looking at (1.1) and having in mind Makarov's result or/and Jones and Wolff solution of Oksendal's conjecture one may conclude that harmonic measure always find some "thin" set of exposed points to concentrate on. This makes plausible the second conjecture that

$$Hdim(\omega) < Hdim J(f) \tag{1.2}$$

for harmonic measure on $J(f)$ unless $J(f)$ is connected. We certainly cannot expect this to happen for arbitrary compact set K. This is clear from the example of Ch.

Bishop [Bi] : for any $\delta < 1$ Bishop constructed a set K such that $Hdim(\omega) = Hdim(K) = \delta$.

However there are many indications that for $K = J(f)$ the conjecture is correct. First of all Zdunik proved that Hausdorff dimension of maximal measure is strictly less than Hausdorff dimension of $J(f)$ unless $J(f)$ is connected for polynomials f. Now Brolin's result shows that (1.2) is true for polynomial dynamics f. For various types of GPL f (1.2) was shown in [MV], [Vo1], [Vo2]. Finally let us mention an interesting result of [Ba] which is in the same vein.

In this paper we are going to prove (1.2) when f is a parabolic Blaschke product. It is worthwhile to mention that then our assumption

$$J(f) \text{ is disconnected}$$

has an ergodic theory interpretation. The fact that $J(f)$ is disconnected means exactly that f acts non-ergodically on the unit circle $\mathbb{T}$ with respect to Lebesgue measure on the circle. In local terms this means that f has only one petal at the parabolic points. The reader can find the discussion of these relationships in Aaranson's papers [A1] and [A2].

Our paper has an analytic part (Section 2), dynamical part (Sections 3,5) and the part where we finally mix analysis and dynamics (Sections 4,6,7). We think that the analytic part is interesting in its own right. The essence of it is a Boundary Harnack Principle (BHP) for a Fatou set of our f. There is an extensive literature on BHP and let us just refer the reader to [An], [Wu] or [JK]. It has been recognized in [Vo2] and [LVo] that BHP may play an important part in metric estimates of harmonic measure on discontinuous fractals. But in all these works mentioned above the existence of BHP relied upon the fact that the domains under consideration have good geometric localization. They are NTA, John or Lipschitz domains. The Fatou set of a parabolic Blaschke product with one petal does not have any nice localization. The complementary intervals of the Julia set are too small with respect to the distance to the parabolic point (see Appendix). However Section 2 contains a certain BHP which is one of the key points in proving (1.2). Another key place is Section 4 where we use [DU3] extensively. We couple here our BHP and the technique of [DU3]. After this we prove that harmonic measure and δ-packing measure on $J(f)$ ($\delta \stackrel{def}{=} Hdim(J(f))$) are singular. Here one comes to an amusing contradiction: if they are not singular then f can be linearized simultaneously in a common neighborhood of different repelling periodic points and the parabolic fixed point So these measures are in fact singular. But this is a much weaker statement then (1.2). To finish the proof we need a third key consideration which amounts to thermodynamical formalism for certain countable state systems with potentials which can be unbounded. Our potentials are of very special kind and this enables us to adapt certain results of [Bo] to our case. This is done in Section 5. After that (1.2) follows easily from the results of this section and the singularity of harmonic and packing measure.

We are in the position to state our main result. Let $\mathbb{D}$ denote the unit disc.The holomorphic coverings $\mathbb{D} \longrightarrow \mathbb{D}$ of finite degree d are called Blaschke products of degree d. By simple conjugacy we may consider them as coverings $\mathbb{C}_+ \longrightarrow \mathbb{C}_+$ that fix ∞. We will freely use this two representations. We consider only Blaschke products with parabolic fixed point. As a function with positive imaginary part in $\mathbb{C}_+$, a Blaschke product of degree d which fixes infinity can be written as

$$f(z) = z + c_0 - \sum_{k=1}^{d-1} \frac{c_k}{z - x_k}$$

where $c_0 \in \mathbb{R}$, $c_k > 0$ for $k = 1, \ldots d-1$. Then f has a petal or two at the parabolic point $p = \infty$ depending on whether $c_0 \neq 0$ or $c_0 = 0$. The former case happens if and only if $J(f) \underset{\neq}{\subset} \mathbb{R}$.

The Main Result Let f be a Blaschke product such that its Julia set $J(f)$ has a parabolic point with just one petal (then the Julia set $J(f)$ is a disconnected subset of the unit circle). Let ω be the harmonic measure in $\bar{\mathbb{C}} \setminus J(f)$. The following holds:

$$Hdim(\omega) < Hdim(J(f)) \quad \square$$

The main analytic tools used to construct and compare the harmonic and the δ-conformal measures are the following two results:

Theorem (Boundary Harnack Principle) If u ,v are positive harmonic functions in Ω , vanishing continuously on some uniformly perfect set K contained in $\mathbb{R}$, and satisfying

$$u(z) = u(\overline{z}) \text{ and } v(z) = v(\overline{z})$$

then the function $\log \frac{u}{v}$ is Hölder continuous of order $\alpha = \alpha(\text{unif.perf})$ on $\Omega \bigcup K$. $\square$

Lemma (On harmonic rigidity) Let u, v be two non-negative subharmonic functions in a disc B with diameter I. Let $J \subset I$ be a closed, uniformly perfect set with infinitely many components.

If u, v vanish on J and are positive and harmonic in $B \setminus J$, and if

$$\frac{u(x)}{v(x)} = \lim_{z \to x} \frac{u(z)}{v(z)} = |A(x)|^2 \; , \; \forall x \in J \tag{1.3}$$

for some holomorphic function A in the ball B , then $|A| \equiv$ constant. $\square$

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2 The Main Lemma.

Definition 2.1 A compact set K is uniformly perfect if

$$\mathrm{cap}(K_f \bigcap B(x,r)) \geq c\ \mathrm{cap}(B(x,r))\ \forall x \in K_f \text{ and } \forall r \leq r_0$$

Through the section 2 we are going to work with domains Ω of the form $\Omega \stackrel{def}{=} B(0,R) \setminus K$ for some uniformly perfect compact set $K \subset \mathbb{R}$ and for some ball $B(0,R)$ with $\mathrm{dist}(K, \partial B(0,R)) \geq \mathrm{diam} K$.
We will denote by $\Omega_+ \stackrel{def}{=} \Omega \bigcap \{z :\ \Im z \geq 0\}$ and by $\Omega_- \stackrel{def}{=} \Omega \bigcap \{z :\ \Im z \leq 0\}$.Given a point $\xi \in K$ we will denote by $\xi^{r+} \stackrel{def}{=} \xi + ir/2$ and by $\xi^{r-} \stackrel{def}{=} \xi - ir/2$.

We are going to deduce a couple of Harnack-type results in Ω_+. Some of them are inspired from [JK] . Most of the constants M that appear depend only on the constant c from the definition (2.1) of uniformly perfect sets ; when this is the case we will write $M = M(\text{unif.perf})$

Proposition 2.2 There exists a universal constant M_1 and some positive ϵ_0 , such that for any $\epsilon \leq \epsilon_0$ and any two points $z_1, z_2 \in \Omega_+$ satisfying $\mathrm{dist}(z_1,z_2) < 2^k\epsilon$ and $\mathrm{dist}(z_i, \partial\Omega_+) > \epsilon$ can be joined by a chain of at most $M_1 k$ balls $B(c_1,r_1), \ldots, B(c_{k'}, r_{k'})$ contained in Ω and having

$$r_p \geq \frac{1}{M_1}\mathrm{dist}(z_i, B(c_p,r_p)) \text{ and } r_p \geq \mathrm{dist}(B(c_p,r_p), \partial\Omega)\ \Box$$

Proposition 2.3 There exists some universal constant M_2 such that for any function u positive and harmonic in Ω and for any $z_1, z_2 \in \Omega_+$ satisfying $\mathrm{dist}(z_1,z_2) < 2^k\epsilon$ and $\mathrm{dist}(z_i, \partial\Omega) > \epsilon$, the following holds:

$$\frac{u(z_1)}{M_2^k} \leq u(z_2) \leq M_2^k\, u(z_1) \quad \Box$$

Proof. Use k times Harnack's principle. $\square$

Remark 2.4 For any M_1 there exists M_2 such that if

$$\frac{u(z_1)}{u(z_2)} \geq M_2^k \quad \text{and } \operatorname{dist}(z_1, z_2) < M_1^k \epsilon \tag{2.1}$$

for some $z_1, z_2 \in \Omega_+$ then $\min_{i=1,2} \operatorname{dist}(z_i, \partial\Omega) < \epsilon$ $\square$

Lemma 2.5 There exists $M_3 = M_3(\text{unif.perf})$ such that for all $q_0 \in K$ and for any u positive harmonic function in Ω such that u vanishes continuously on $B(q,r) \bigcap K$ the following holds:

$$\sup\{u(x);\ x \in B(q_0, s/M_3)\ \} \ \leq \frac{1}{2} \sup\{u(x)\ ; x \in B(q_0, s)\} \ \text{ for all } s < r \quad \square$$

Proof: Denote by $a = \sup\{u(x),\ x \in B(q_0, s)\}$. Then

$$u(x) \leq a\ \omega_{B(q_0,s)\setminus K}(\partial B(q_0, s), x)$$

By the uniformly perfectness of K there exists some $\gamma = \gamma(\text{unif.perf})$ such that

$$\omega_{B(q_0,s)\setminus K}(\partial B(q_0, s), x) \leq \gamma < 1 \text{ for all } x \in B(q_0, s/2).$$

Let $M_3 = 2^n$ for some n large such that $\gamma^n \leq \frac{1}{2}$. Let $z \in B(q, s/M_3)$. Then

$$u(z) \leq \gamma^n a \leq \frac{1}{2} \sup\{u(x),\ x \in B(q_0, s)\} \quad \square$$

Lemma 2.6 There exists $M_4 = M_4(\text{unif.perf})$ such that if $q_0 \in K$ and if u is a positive harmonic function in Ω that vanishes continuously on $K \bigcap B(q_0, 2r)$ then the following holds:

$$u(x) \leq M_4\ [\ u(q_0^{r+}) + u(q_0^{r-})\ ] \ \text{ for all } x \in B(q_0, r) \quad \square$$

Proof. Let $x \in \Omega_+$. Let M_3 as in Lemma 2.5 . Let M_2 satisfy:

$$\frac{u(z_1)}{u(z_2)} > M_2^k \text{ and } \operatorname{dist}(z_1, z_2) < \epsilon \Rightarrow \min \operatorname{dist}(z_i, K) < \frac{\epsilon}{M_3^k}$$

Let N be large such that $2^N > M_2$. Claim that $u(x) \leq M_2^{N+4}\ u(q_0^{r+})$ for all $x \in B(q_0, r) \bigcap \Omega_+$.

If not, $\exists x_1 \in B(q_0, r)$ such that $u(x_1) \geq M_2^{N+4}\ u(q_0^{r+})$. By

$$\frac{u(x_1)}{u(q_0^{r+})} \geq M_2^{N+4} \text{ and } \operatorname{dist}(x_1, q_0^{r+}) < 2r$$

we have $\operatorname{dist}(x_1, K) \leq \frac{2r}{M_3^{N+4}}$. Let q_1 be a point in K closest to x_1. Then $\operatorname{dist}(q_1, q_0) < r + \frac{4r}{M_3^{N+4}}$. By Lemma 2.5

$$\begin{aligned}\sup\{u(x); x \in B(q_1, r/M_3^3)\} \quad &\geq 2^N \ \sup\{u(x); x \in B(q_1, r/M_3^{N+3})\} \\ &\geq 2^N u(x_1) \geq M_2 u(x_1) \geq M_2^{N+5} \ u(q_0^{r+}) \end{aligned} \tag{2.2}$$

In this manner we can find $x_2 \in B(q_1, r/M_3^3)$ such that $u(x_2) \geq M_2^{N+5} u(q_0^{r+})$

By (2.1), $\operatorname{dist}(x_2, K) \leq \frac{4r}{M_3^{N+5}}$. Let q_2 be a point in K closest to x_2. Then

$$\operatorname{dist}(q_2, q_0) \leq r + \ 4r/M_3^{N+4} + \ r/M_3^3 + \ 4r/M_3^{N+5} \ \leq 2r$$

and

$$\begin{aligned}\sup\{u(x), \ x \in B(q_2, r/M_3^4\} \quad &\geq 2^N \sup\{u(x), \ x \in B(q_2, r/M_3^{N+4})\} \\ &\geq 2^N u(x_2) \geq M_2^{N+6} u(q_0^{r+}) \end{aligned} \tag{2.3}$$

Finally we can find a sequence of points $x_1, x_2, \ldots$ such that $x_n \to K$ with $\operatorname{dist}(x_n, q_0) < 2r$ and $u(x_n) \geq M_2^{N+n} u(q_0^{r+}) \to \infty$. This is impossible because u vanishes continuously on $B(q_0, 2r)$. Therefore the claim is true.

Theorem 2.7 If u ,v are positive harmonic functions in Ω , vanishing continuously on K, and satisfying

$$u(z) = u(\overline{z}) \text{ and } \ v(z) = v(\overline{z})$$

then the function $\log \frac{u}{v}$ can be extended to a Hölder continuous function of order $\alpha = \alpha(\text{unif.perf})$ on $\Omega \bigcup K$. $\square$

Proof. Let $q \in K$ and $r > 0$ be fixed. Denote by $\Gamma_n \stackrel{def}{=} \partial B(q, r/4^n)$ and by $\Gamma_{n+1/2} \stackrel{def}{=} \partial B(q, r/4^{n+1/2})$. Following [LV] it is enough to show that

$$1 = \min_{\Gamma_n} \frac{u}{v} \ \leq \ \max_{\Gamma_n} \frac{u}{v} \ \leq 1 + \epsilon_n \quad \Rightarrow \quad \frac{\max_{\Gamma_{n+1}} \frac{u}{v}}{\min_{\Gamma_{n+1}} \frac{u}{v}} \ \leq \ 1 + \beta \epsilon_n \tag{2.4}$$

for some $\beta = \beta(\text{unif.perf}) < 1$.

Decompose Γ_n into

$$\Gamma_n^L = \{1 + \frac{\epsilon_n}{2} < \frac{u}{v} \leq 1 + \epsilon_n\} \tag{2.5}$$

and

$$\Gamma_n^S = \{1 \leq \frac{u}{v} \leq 1 + \frac{\epsilon_n}{2}\} \tag{2.6}$$

We are going to use the following harmonic measures:

$$\omega(\cdot) = \omega_\Omega(\cdot, \infty)$$

$$\omega_{\Gamma_n}(\cdot,\xi) = \omega_{B(q,r/4^n)\backslash K}(\cdot,\xi) \quad \text{for } \xi \in B(q, r/4^n)$$

$$\omega_{n+\frac{1}{2}}(\cdot,\xi) = \omega_{B(q,r/4^{n+1/2})\backslash K}(\cdot,\xi) \quad \text{for } \xi \in B(q, r/4^{n+1/2})$$

By Poisson's formula we get:

$$u(\xi) = \int_{\Gamma_n^L} u(\eta) d\omega_{\Gamma_n}(\eta,\xi) \; + \int_{\Gamma_n^S} u(\eta) d\omega_{\Gamma_n}(\eta,\xi)$$

Denote by $v^L(\xi) = \int_{\Gamma_n^L} v(\eta) d\omega_{\Gamma_n}(\eta,\xi)$ and by $v^S(\xi) = \int_{\Gamma_n^S} v(\eta) d\omega_{\Gamma_n}(\eta,\xi)$

Then $v^S \; + \; (1+\epsilon_n/2)\, v^L \; \le u(\xi) \le (1+\epsilon_n/2)\, v^S \; + \; (1+\epsilon_n)\, v^L$

Let $\xi_+ \in \Gamma_{n+1/2}$ be $\xi_+ = q + ir/4^{n+1/2}$ and $\xi_- \in \Gamma_{n+1/2}$ be $\xi_- = q - ir/4^{n+1/2}$

Suppose that

$$v^S(\xi_+) \ge v^L(\xi_+) \tag{2.7}$$

Then the same is true for ξ_-.

Claim: There exists $c = c(\text{unif.perf})$ such that

$$v^S(\xi) \ge c\, v^L(\xi) \text{ for all } \xi \in \Gamma_{n+1}. \tag{2.8}$$

Assume the claim is true. Then by (2.5) and by $v^L + v^S = v$ we get

$$u(\xi) \le (1+\epsilon_n/2)\, v^S(\xi) \; + (1+\epsilon_n)\, v^L(\xi) \; \le (1+\epsilon_n/2)\, v(\xi) \; + \; \epsilon_n/2\, v^L(\xi)$$

therefore

$$u(\xi) \le (1+\epsilon_n/2)\, v(\xi) \; + \epsilon_n/2 \, \frac{1}{1+c}\, v(\xi) \; \le \; (1+\beta\epsilon_n)\, v(\xi)$$

and

$$\frac{\max_{\Gamma_{n+1}} \frac{u}{v}}{\min_{\Gamma_{n+1}} \frac{u}{v}} \; \le \; \frac{1+\beta\epsilon_n}{\min_{\Gamma_n} \frac{u}{v}} \; = (1+\beta\epsilon_n)$$

If the opposite of (2.7) is true, one can prove that the opposite of the inequality (2.8) holds and the minimum of $\frac{u}{v}$ can be estimated from below in order to deduce (2.4).

Proof of the claim: Let $\xi \in \Gamma_{n+1}$. For convenience suppose that $q=0$. Decompose $\Gamma_{n+1/2}$ into four arcs

$$I_+ = (e^{i\pi/20}, e^{i19\pi/20})\frac{r}{4^{n+1/2}}, J- = (e^{i19\pi/20}, e^{i21\pi/20})\frac{r}{4^{n+1/2}},$$

$$I- = (e^{i21\pi/20}, e^{i39\pi/20})\frac{r}{4^{n+1/2}}, J_+ = (e^{i39\pi/20}, e^{i\pi/20})\frac{r}{4^{n+1/2}}.$$

The arcs I_+ and I_- are far from the boundary of Ω. By Harnack principle $v(\eta)$ and $v(\xi_+)$ are comparable for $\eta \in I_+$. Therefore

$$v^S(\xi) \geq \int_{I_+ \bigcup I_-} v^S(\eta) d\omega_{n+\frac{1}{2}}(\eta,\xi) \ \geq c_1 \ v^S(\xi_+) \ \omega_{n+\frac{1}{2}}(I_+ \bigcup I_-, \xi) \tag{2.9}$$

and

$$v^L(\xi) \leq c_2 \ v^L(\xi_+) \ \omega_{n+\frac{1}{2}}(I_+ \bigcup I_-, \xi) \ + \ \int_{J_+ \bigcup J_-} v^L(\eta) d\omega_{n+\frac{1}{2}}(\eta,\xi) \tag{2.10}$$

The estimate for v^L will be done in two steps:

1. Prove that $v^L(\xi) \leq c' \ v^S(\xi_+)$ for $\xi \in J_+ \bigcup J_-$.
2. Prove that $\omega_{n+\frac{1}{2}}(J_+ \bigcup J_-, \xi) \leq c' \ \omega_{n+\frac{1}{2}}(I_+ \bigcup I_-, \xi_+)$ for $\xi \in \Gamma_{n+1}$.

Once the two steps are concluded the proof of the claim becomes trivial.

Step 1. Consider the smallest disc that contains J_+. If it doesn't meet K there is nothing to prove. If it contains a point q_0 of K, apply Lemma 2.6 for the function v^L and the disc $B(q_0, 2\mathrm{diam}(J_+))$. We get that

$$v^L(\xi) \leq M_4[\ v^L(q_0^{r+}) + v^L(q_0^{r-})] \ \leq M_4'[\ v^S(q_0^{r+}) + v^S(q_0^{r-})\ ], \ \ \forall \xi \in J_+ \tag{2.11}$$

By the Harnack principle $v^S(q_0^{r+})$ and $v^S(\xi_+)$ are comparable.

Step 2. Let $G_{n+\frac{1}{2}}$ be the Green function in $B(q, r/4^{n+1/2}) \setminus K$. Denote by c_J the middle point of J_+; let ϕ be a smooth function such that

$$\phi = 1 \text{ on } \ B(c_J, |J_+|) \text{ and } \quad \phi = 0 \quad \text{outside } \ B(c_J, 2|J_+|)$$

We get:

$$\begin{aligned} \omega_{n+\frac{1}{2}}(J_+,\xi) \leq \ & \textstyle\int_{B(c_J,|J_+|)} \Delta G_{n+\frac{1}{2}}(z,\xi)dz \leq \int_{B(c_J,2|J_+|)} \phi(z)\Delta G_{n+\frac{1}{2}}(z,\xi)dz \\ & = \textstyle\int_{B(c_J,2|J_+|)} \Delta\phi(z) G_{n+\frac{1}{2}}(z,\xi)dz \end{aligned} \tag{2.12}$$

Recall that ξ is a point in Γ_{n+1} . Apply Lemma 2.6 to the harmonic function $G_{n+\frac{1}{2}}(\cdot,\xi)$ and the ball $B(c_J,2|J_+|)$ in the domain $B(c_J,10|J_+|)$. To simplify notations let $p_+ = c_J + i\frac{|J_+|}{2}$ and $p_- = c_J - i\frac{|J_+|}{2}$

We get:

$$G_{n+\frac{1}{2}}(z,\xi) \le M_5\,[G_{n+\frac{1}{2}}(p_+,\xi) + G_{n+\frac{1}{2}}(p_-,\xi)] \text{ for all } z \in B(c_{+J},2|J_+|) \tag{2.13}$$

Use (2.13) and the fact that the function ϕ has $|\Delta\phi| \le \frac{\text{const}}{|J_+|^2}$ in the inequality (2.12). We get

$$\omega_{n+\frac{1}{2}}(J_+,\xi) \le M_5\,[G_{n+\frac{1}{2}}(p_+,\xi) + G_{n+\frac{1}{2}}(p_-,\xi)] \tag{2.14}$$

Comparing $G_{n+\frac{1}{2}}(\cdot,p_+)$ and $\omega_{n+\frac{1}{2}}(I_+\bigcup I_-,\cdot)$ on the domain $B(q_0,\frac{r}{4^{n+1/2}}) \setminus B(p_+,\frac{r}{100\,4^{n+1/2}}) \setminus K$ we get that

$$G_{n+\frac{1}{2}}(\xi,p_+) \le \text{const}\ \omega_{n+\frac{1}{2}}(I_+\bigcup I_-,\xi) \text{ for all } \xi \in \Gamma_{n+1} \tag{2.15}$$

The last inequality and (2.14) conclude the proof of Step 2. $\Box$

Corollary 2.8 If $J_f \subset \mathbb{R}$ is the Julia set of a rational map and if G is the Green function in $\mathbb{C}\setminus J_f$, then the function $\log\frac{G(f(z))}{G(z)}$ can be extended to J_f as a Hölder continuous function.

3 Jump Transformation

In order to compare the harmonic measure and the Hausdorff measure we would like to construct some f-invariant measures equivalent to each of them. If we attempt to construct a finite f-invariant measure equivalent to ω, a necessary condition is :

$$\text{if a set } A \subset J \text{ has } \omega(A) < 1 \text{ then } \omega(f^{-n}A) < 1-\epsilon < 1.$$

By $\omega(f^{-n}A,x) = \omega(A,f^n(x) \to 1$ as $n \to \infty$, there is no finite f-invariant measure equivalent to ω.

Given a parabolic Blaschke product f with one petal and the harmonic measure ω on J we will construct a transformation T on an open, dense subset $J_\star$ of the Julia set, which we will call jump transformation, such that the triplet $(\omega,T,J_\star)$ satisfies Walters' conditions [Wa].

First of all ω is going to be totally non-singular:

$$\omega(E) = 0 \Longleftrightarrow \omega(TE) = 0 \tag{3.1}$$

and the transformation will satisfy:

$$\exists \epsilon_0 > 0 \ \text{ such that } \ \forall x \in J_\star, \ T^{-1}(B_{2\epsilon_0}(x) \bigcap J_\star) = \sqcup_i \ A_i(x), \tag{3.2}$$

that is the disjoint union of open sets such that $T : A_i \to B_{2\epsilon_0}(x) \bigcap J_\star$ is a homeomorphism and $\text{dist}(Ty, Ty') \geq \text{dist}(y, y') \quad \forall y, y' \in A_i(x)$.
Also

$$\forall \epsilon > 0 \ \exists M \ \text{ such that } \ \forall x \in J_\star, \ T^{-M} \ \text{ is } \ \epsilon\text{-dense in } \ J_\star \tag{3.3}$$

Notice that (3.1) implies the fact that the push-forward measure $T^\star \omega = \omega \circ T^{-1}$ is absolutely continuous with respect to ω. Also the Jacobian of ω with respect to T is well defined and we will introduce the notation

$$\psi = -\log \frac{d\omega \circ T}{d\omega}(x) = -\log J_\omega^T(x)$$

To write down the rest of Walters' conditions we need more notations: let $S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(T^i x)$ for any function $\varphi : J_\star \to \mathbb{R}$. Given b an element of $T^{-n}b_0$ for some $b_0 \in J_\star$, let T_b^{-n} denote the inverse branch that sends $b_0 \to b$. Set $c_{\varphi,b}(x, x') = |S_n\varphi(y) - S_n\varphi(y')|$ for $y = T_b^{-n}x, y' = T_b^{-n}x'$.

Let $\mathcal{F}_T$ be the class of functions φ such that

(i) $\quad \sum_{y \in T^{-1}x} e^{\varphi(y)} \leq K_\varphi < \infty \quad$ (3.4)

(ii) $\quad c_\varphi(x, x') = \sup_{n \geq 1} \sup c_{\varphi,b}(x, x') \leq C_\varphi < \infty$ for x,x' close enough $\quad$ (3.5)

(iii) $\quad c_\varphi(x, x') \to 0 \ \text{ as } \ d(x, x') \to 0 \quad$ (3.6)

Remark 3.1 If φ is the Jacobian of the harmonic measure with respect to jump transformation, then

(3.4) $\Leftrightarrow$ the density of $\omega \circ T^{-1}$ with respect to ω is bounded

(3.5) $\Leftrightarrow$ $\sup_n \sup\{\frac{J_\omega^{T^n}(y)}{J_\omega^{T^n}(y')}; y \in T^{-n}x, y' \in T^{-n}x'\} \leq A(x, x') \leq K < \infty$

(3.6) $\Leftrightarrow$ $A(x, x') \to 1$ when $d(x, x') \to 0$

Let $UC(X)$ denote the space of uniformly continuous functions on some set X. For $\varphi \in \mathcal{F}_T$ define the operator $\mathcal{L}_\varphi$ on $UC(X)$:

$$\mathcal{L}_\varphi g(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} g(y)$$

Recall that we have denoted by δ the Hausdorff dimension of J. We are going to show that both potentials $\psi = -\log J_\omega^T$ and $\phi = -\delta \log |T'|$ satisfy Walters' conditions (3.4) to (3.6) so we will be able to apply the following theorem:

Theorem 3.1 ([Wa]) Suppose $T : X_0 \to X_0$ satisfies (3.2) and (3.3) and that $\varphi \in \mathcal{F}_T$. Then $\mathcal{L}_\varphi$ extends to a bounded linear operator in $C(X_0)$.

Furthermore, there exists an unique probability measure μ satisfying (3.1) and some $\lambda > 0$ such that

$$\mathcal{L}^\star_\varphi \mu = \lambda \, \mu \tag{3.7}$$

There exists a unique positive function $h, \ h \in C(\bar{X}_0)$ such that

$$\mathcal{L}_\varphi h = \lambda h \quad \text{and} \quad \mu(h) = 1 \tag{3.8}$$

The measure $h\mu$ is T-invariant. $\square$

See [Wa] for details.

This could be enough in order to construct invariant measures equivalent to the δ-conformal measure and equivalent to the harmonic measure. However, we need better properties of the densities h in equation (3.8). For this reason the actual construction of the invariant measures will be done in the next section, using a theorem of Ionescu Tulcea.

Following M.Denker and M.Urbanski, [DU1], [DU2], [DU3], we will construct the Jump transformation T for conformal parabolic systems. For the sake of completeness we are obliged to repeat parts of these works.

Let J be a compact set consisting of at least two points, let $J \subset U \subset \mathbb{C}$ be an open neighborhood of J and let $f : U \to \mathbb{C}$ be a holomorphic map. We assume that there exists a point p such that $f(p) = p$ and $f'(p) = 1$.

Definition 3.2 A system (f, U, J) is called conformally parabolic if there exists a unique parabolic point p in U and if

(i) $f(J) = J$

(ii) there are no critical points in J

(iii) there exists a neighborhood W of p such that if $x \in U$ satisfies $f^n(x) \in W$ for $\forall n > 0$ then $x = p$

(iv) for $x \in U \setminus J$, $\exists r_x > 0$ such that $f^n(B(x, r_x)) \to p$ uniformly

(v) if $f^n(x) \in U, \forall n > 0$ then $x \in J$ or $f^n(x) \to p$

(vi) the mapping $f : J \to J$ is topologically exact.

Lemma 3.2 Let V be a open neighborhood of the parabolic point p. There exists $\delta_0 = \delta_0(V) > 0$ such that for every $z \in J \setminus V$ all inverse branches of f^n are well defined on $B(z, 3\delta_0)$. $\Box$

Proof: See [DU3].

Let now f be a parabolic Blaschke product with Cantor like Julia set, having the parabolic point at $p = 1$. Denote by Γ^α the cone domain with vertex at the parabolic point, that is $\Gamma^\alpha =$ convh$(1, B(0, \sin\alpha))$.

Lemma 3.3 There exists α such for every $z \in J$ all inverse branches of f^n are well defined in $B(z, r)$ if $B(z, r) \bigcap \Gamma^\alpha = \emptyset$ $\Box$

Lemma 3.4 Such a transformation f is positively expansive, i.e.

$$\exists\ \delta' > 0 \text{ such that if } \sup_{n>0} d(f^n(x), f^n(y)) \le \delta' \text{ then } x = y.$$

Proof: See [DU1].

Recall that a cover $\mathcal{R} = \{R_1, R_2, \dots R_s\}$, $s \le \infty$ of X is said to be a Markov partition if it satisfies:

(i) $R_i = \overline{\text{int} R_i}$

(ii) $\text{int} R_i$ are disjoint

(iii) $f(R_i)$ is the union of some R_j.

Assume that the map $F : X \to X$ is open, surjective and positively expansive. From the proofs in Ruelle's book [Ru] the following result can be deduced:

Theorem 3.5 If $F : X \to X$ is an open, surjective, positively expansive continuous mapping of a compact metric space X, and if μ is an atomless measure, then there exists a finite Markov partition of arbitrary small diameters satisfying: $\mu(\partial R_1 \bigcup \partial R_2 \bigcup \dots \partial R_s) = 0$. $\Box$

The transition matrix $A = (A_{i,j})_{i,j \le s}$ associated with the Markov partition $\mathcal{R}_0$ is defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } F(R_i) \supset R_j \\ 0 & \text{if } F(R_i) \bigcap R_j = \emptyset \end{cases}$$

A sequence $k_1, \dots, k_n$ is said to be A-admissible if $A_{k_i k_{i+1}} = 1$ for every $i = 1, \dots n-1$. Coming back to our parabolic system, given a Markov partition and some A-admissible sequence $k_1, \dots, k_n$ define $A(k_1, \dots, k_n) = \bigcap_{j=1}^n f^{-j+1} R_{k_j}$. We will call such a set an n-cylinder. The family of all n-cylinders will be denoted $\mathcal{R}_0^n$. Among all cylinders of $\mathcal{R}_0^n$ we distinguish "good" and "bad" cylinders. A cylinder will be called good if $f^{n-1}A(k_1,\dots k_n)\bigcap W=\emptyset$, and bad cylinder if $f^{n-1}A(k_1,\dots k_n)\bigcap W\neq\emptyset$. The set W is the neighborhood of the parabolic point from Definition 3.2. Let $\mathcal{G}_n$ be the collection of the good cylinders of order n, let $\mathcal{B}_n$ be the collection of the bad cylinders of order n, and $\mathcal{R}_{\mathcal{G}}$ be the collection of all good cylinders, $\mathcal{R}_{\mathcal{B}}$ be the collection of all bad cylinders.

Remark 3.6 Since the map f is expansive and so expanding with respect to a metric compatible with the topology on J, (see theorem 2.2 of [DU2]), the diameters of elements of $\mathcal{R}_0^n$ tend to zero and so the family $\mathcal{R}_0$ generates the Borel σ-algebra mod ν for any non-atomic measure ν on J.

Lemma 3.7 Let ν be a non-atomic measure on J. Then $X_0 \stackrel{def}{=} \bigcup_{A\in\mathcal{R}_G} A$ satisfies $\nu(X_0)=1$ and

$$\lim_{n\to\infty}\Sigma_{A\in\mathcal{B}_n}\nu(A)=0 \quad \Box$$

See [DU3] for proof.

Now we are ready to introduce the jump transformation of f.

For every $x\in X_0$ take the smallest rank of a good n-cylinder that contains x, i.e.

$$N(x)=\inf\{n\,;x\in A(k_1,\dots k_n) \text{ and } f^{n-1}A(k_1,\dots k_n)\bigcap W=\emptyset\}.$$

In view of Lemma 3.7, $N(x)$ is almost everywhere finite with respect to any non-atomic measure. Schweiger's jump transformation is defined by

$$T(x)=f^{N(x)}(x).$$

The set $N^{-1}(n)$ consists exactly of good cylinders $A(k_1,\dots k_n)\in\mathcal{G}_n$ that have a "bad" parent: $A(k_1,\dots k_{n-1})\in\mathcal{B}_{n-1}$.

Let us consider a new family, $\mathcal{R}_\star$, to be the union for all n of families of cylinders forming $N^{-1}(n)$. Denote the elements of $\mathcal{R}_\star$ by C_i. Then Lemma 3.7 shows that

$$\bigcup_{C\in\mathcal{R}_\star} C=X_0 \text{ and } T(C)=\bigcup_{C'\in\mathcal{R}_\star, C'\bigcap T(C)\neq\emptyset} C' \tag{3.9}$$

In this sense $\mathcal{R}_\star$ constitutes a countable Markov partition for T. Also the transition matrix for $(T,\mathcal{R}_\star)$ is aperiodic.

For $x\in X_0$ define the positive integers $N_n(x)$ by:

$$N_1(x)=N(x)$$

$$N_{n+1}(x)=N_n(x)+N(T^n(x))$$

which gives $T^n(x) = T^{N_n(x)}(x)$. If we denote by $C^n(x)$ the cylinder of $\mathcal{R}_\star^n$ containing x , then also

$$T^n_{|C^n(x)} = f^{N_n(x)}_{|C^n(x)}$$

The following result is proved in [DU3]:

Lemma 3.8 The jump transformation is expanding, i.e. there exist $c, \beta > 0$ such that

$$|(T^n)'(x)| \geq c\, e^{\beta n} \text{ for all } x \in X_0, n > 0 \ \square$$

Lemma 3.9 There exist constants B, η depending only on f, such that for all pairs of points x, x' sufficiently close, if $y = T_b^{-n}x,\ y' = T_b^{-n}x'$, then $\forall n$

$$\frac{|(T^n)'(y')|}{|(T^n)'(y)|} \leq e^{\frac{B}{\eta}d(x,x')} \quad \square \tag{3.10}$$

Let $\Sigma_\star \subset \mathcal{R}_\star^{\mathbb{N}}$ be the space of all admissible sequences of cylinders according to the transition matrix of $(T, \mathcal{R}_\star)$.
Let $\Sigma_\star^0 = \{(C_n)_{n\geq 0} \in \Sigma_\star$ such that $\bigcap_n T^{-n}(C_n) \neq \emptyset\}$. The spaces $\Sigma_\star$ and $\Sigma_\star^0$ may not be compact. Since by lemma 3.8 the diameters of of the partition $\mathcal{R}_\star^n$ converge to zero, one can define the projection $\pi : \Sigma_\star^0 \to X_0$ setting

$$\pi(\ (C_n)_{n\geq 0}\ ) = \bigcap_{n\geq 0} T^{-n}(C_n)$$

Consider the metric ρ on $\Sigma_\star^0$ to be

$$\rho((C_n)_{n\geq 0}, (C'_n)_{n'\geq 0}) \ = \ e^{-\min\{n;\, C_n \neq C'_n\}}$$

Proposition 3.10 The map $\pi : \Sigma_\star^0 \to X_0$ is a Hölder continuous surjection, with exponent β from Lemma 3.8 $\square$

Remark 3.11 If we identify $x \in X_0$ with $\pi^{-1}x$ then one can transport the metric ρ to X_0. Previous lemma can be formulated as

$$\forall x, x' \in X_0 \quad |x - x'| = d(x, x') \leq c\rho(x, x')^\beta$$

Using the fact that for any cylinder $C \in \mathcal{R}_\star$ there exists a cylinder $A \in \mathcal{R}_0$ such that

$$T(C) = f(A), \tag{3.11}$$

one can prove the following lemma:

Lemma 3.12 1. For any $\epsilon > 0$ there exists some M such that $\forall x \in X^0$, $T^{-M}x$ is ϵ-dense in (X_0, ρ)
2. For any $\epsilon > 0$ there exists some M such that $\forall x \in X_0$, $T^{-M}x$ is ϵ-dense in (X_0, d) $\square$

Using Lemma 3.4 we can conclude that

Lemma 3.13 For every $C^{n+1} \in \mathcal{R}_\star^{n+1}$ there exists a unique holomorphic inverse branch T^{-n} of T^n defined on $B(T^n(C^{n+1}), 2\delta_0)$ and sending $T^n(C^{n+1})$ to C^{n+1}. $\square$

Using Köebe distortion properties for the functions T^{-n} and Lemma 3.9, one can show that for points y, y' lying in the same n-cylinder of $\mathcal{R}_\star^n$, the following holds:

$$d(y, y') \le \frac{\text{const}}{\text{diam} R_p} e^{-\beta n} d(T^n y, T^n y') \tag{3.12}$$

Let $\mathcal{C}_i$ denote the family of $\mathcal{R}_\star$ such that $T(C) = f(R_i \bigcap X_0)$, $i = 1, \ldots, s$ (see equation 3.11). In the next two sections we will refer to these cylinders as standard cylinders.

If we select the initial Markov partition to consist of $R_i = J \bigcap I_i$ where each I_i is an arc of the unit circle that is mapped one-to-one and onto the full circle and the first of them, I_1, is the arc containing the parabolic point, then the standard cylinders turn out to be exactly the sets $f_1^{-n} I_j$, $j = 2, \ldots, s$, where f_1^{-1} denotes the inverse branch that sends the unit circle onto I_1.

The set X_0 that we get for this particular choice of cylinders is $J \setminus \bigcup_n f^{-n}(p)$. We will denote it by $J_\star$.

A similar convention will be made if we work with J_f on the real line instead of the circle.

4 Construction of Invariant Measures

Let $\delta = Hdim(J)$. We will first construct an invariant measure whose Hausdorff dimension is equal to δ. To do that we need to introduce conformal measures. Recall that a probability m is said to be δ-conformal with respect to some transformation F if

$$m(F(A)) = \int_A |F'|^\delta dm \quad \text{for all sets A such that } F_{|A} \text{ is one-to-one.}$$

It was proved in [DU2] that for parabolic systems there exists exactly one δ-conformal measure m with respect to the jump transformation. Moreover m is non-atomic.

Lemma 4.1 There exists some $K < \infty$ such that

$$\sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta} \leq K \text{ for all } x \in X_0.$$

Proof: Let us denote $L(x) = \sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta}$. Lemma 3.13 implies that if $y, y' \in C_i$, for some standard cylinder C_i, then

$$B^{-1} \leq \frac{|T'(y)|}{|T'(y')|} \leq B$$

So if $x, x' \in f(R_i \bigcap J_\star)$, then $\frac{L(x)}{L(x')} \leq B$. Use the fact $|T'|^\delta$ is the Jacobian of the δ-conformal measure with respect to T. So

$$\int_{f(R_i \bigcap J_\star)} L(x) dm(x) = \sum_{C \in \mathcal{C}} \int_C |T'(y)|^{-\delta} dm(Ty) = \sum_{C \in \mathcal{C}} m(C) = 1$$

But the integral majorizes the quantity $B \sup_{x \in f(R_i \bigcap J_\star)} L(x) m(f(R_i \bigcap J_\star)$. Thus

$$L(x) \leq B \frac{1}{\min_{1 \leq i \leq s} m(f(R_i \bigcap J_\star))} = K < \infty$$

Note that we can actually show that the convergence in the series $\sum_{y \in T^{-1}x} \frac{1}{|T'(y)|^\delta}$ is uniform in $x \in J_\star$, in the sense that

$$\sup_{x \in J_\star} \sum_{y \in T^{-1}x \bigcap (\bigcup_{i=k+1}^\infty C_i)} \frac{1}{|T'(y)|^\delta} = \epsilon^k \to 0 \text{ as } k \to \infty \tag{4.1}$$

Recall that if two points y, y' are contained in the same cylinder then $c^{-1} \leq \frac{|T'(y)|}{|T'(y')|} \leq c$. That means that as x varies in $J_\star$, the sums in (4.1) differ at most by a factor of c, therefore the convergence is uniform.

Let $\phi(x) = -\delta \log |T'|(x)$, for $x \in J_\star$. The measure m, the transformation T, and the potential ϕ satisfy all Walters' conditions (3.1)-(3.6). We conclude that

Theorem 4.2 There exists a unique T-invariant probability measure μ absolutely continuous with respect to m. Moreover μ is ergodic. $\square$

For the harmonic measure, fix a reference point P_0 outside the Julia set and let $\omega = \omega(\cdot, P_0)$, $G = G(\cdot, P_0)$. We introduce $\mathcal{J}(z) = \frac{G(f(z))}{G(z)}$, $z \in \mathbb{C}$. By Harnack inequality, for z in a certain neighborhood U of J,

$$0 < c \leq \mathcal{J}(z) \leq C < \infty$$

It will be more convenient for us to change coordinates and consider J_f a subset of the real line rather than of the circle, and the parabolic point to be $p = 0$, rather than $p = 1$.

Lemma 4.3 1. For all $x \in J_f$ there exists the limit

$$\lim_{z \to x, z \in U \setminus J_f} \mathcal{J}(z) \stackrel{def}{=} \mathcal{J}(x)$$

2. For $\omega - a.e.\ x \in J_f$, $\mathcal{J}(x) = Jac^f_\omega(x)$. $\square$

Proof: the first part of this lemma follows from Lemma 2.8. For the second part we will use the following result from [Gr] and [F]:

Grishin's Lemma Let V be a neighborhood of a compact K and let u, v be two subharmonic functions on V, vanishing on K and positive and harmonic in $V \setminus K$. Suppose that for every $x \in \partial K$ there exists the limit

$$\rho(x) = \lim_{\xi \to x, \xi \in V \setminus K} \frac{u(x)}{v(x)}$$

and this limit is continuous. Then $d\mu_u = \rho\ d\mu_v$, where μ_u, μ_v are the Riesz measures of u and v. $\square$

Applying it for the functions $u = G \circ f$, $v = G$ whose Riesz measures are $\mu_u = \Delta(G \circ f)$ and $\mu_v = \Delta G = \omega$, we get that for sets $E \subset J_f$ on which $f : E \to f(E)$ is injective we have

$$\omega(f(E)) = \mu_u(E) = \int_E \mathcal{J}(x) d\mu_v(x) = \int_E \mathcal{J}(x) d\omega(x)$$

therefore $\mathcal{J}$ is the Jacobian of the harmonic measure. In what follows we mean by $Jac^f_\omega(x)$ this continuous representative $\mathcal{J}(x)$. Let $Jac^{T^n}_\omega$ be the Jacobian of the harmonic measure with respect to T^n.

Lemma 4.4 Let $C(k_1, \ldots k_n)$ be a n-cylinder of $\mathcal{R}_\star$. The map $T^n : C(k_1, \ldots k_n) \to X_0$ is injective and if $y \in T^{-n}x$, $y' \in T^{-n}x'$ are two points of the same n-cylinder, then

$$\left| \frac{Jac^{T^n}_\omega(x)}{Jac^{T^n}_\omega(x')} - 1 \right| \leq\ c\,\mathrm{dist}(x, x')^\beta \ \square$$

Proof: Let g be a holomorphic inverse branch of T^n defined on $B(fR_i, 2\delta_0) = B(C_{k_n}, 2\delta_0)$ such that $g(T(C_{k_n})) = C(k_1, \ldots k_n)$. Thus $g(x) = y$, $g(x') = y'$. Such a branch exists by Lemma 3.13. We have to estimate the following ratio:

$$\frac{\mathcal{J}(y)\mathcal{J}(Ty)\ldots\mathcal{J}(T^{n-1}y)}{\mathcal{J}(y')\mathcal{J}(Ty')\ldots\mathcal{J}(T^{n-1}y')} \tag{4.2}$$

Let us choose points $\zeta,\ \zeta'$ in $U \setminus J$ extremely close to y, y', and let us estimate

$$\frac{G(T^n\zeta)G(\zeta')}{G(T^n\zeta')G(\zeta)} \tag{4.3}$$

If we can estimate the previous ratio uniformly in $\zeta \to y,\ \zeta' \to y'$, we get the estimate for the ratio in (4.2).

Now let $z = T^n\zeta,\ z' = T^n\zeta'$. Both points can be assumed to be in $B(C_{k_n}, 2\delta_0)$ as $\zeta \sim y, \zeta' \sim y'$. Rewrite the double ratio in (4.3) as

$$\frac{G(gz')}{G(z')} : \frac{G(gz)}{G(z)} \tag{4.4}$$

The functions $u = G \circ g,\ v = G$ satisfy the conditions of Theorem 2.7, so the latter double ratio has the following estimate with constants independent of $u, v,$ and $z, z' \in B(f(R_i), \delta_0)$:

$$|\frac{u(z')}{v(z')} : \frac{u(z)}{v(z)} - 1| \le c\,[d(z,z')]^\beta$$

Just let $\zeta \to z,\ \zeta' \to z', z \to x,\ z' \to x'$ and we get the desired estimate.

Now to imitate the results of Section 3 of [DU] we need the following notations:

$$\mathcal{H}_0 = \{g;\ g : X_0 \to \mathbb{R} \text{ bounded}\}$$

and

$$\mathcal{H}_\alpha = \{g \in \mathcal{H}_0;\ v_\alpha(g) < \infty\},$$

where $v_\alpha(g) = \sup\{\left|\frac{g(x)-g(x')}{\rho(x,x')^\alpha}\right|,\ x, x' \in J_\star, \rho(x,x') \le e^{-1}\ \}$.

These are Banach spaces, $\mathcal{H}_0$ with the supremum norm and $\mathcal{H}_\alpha$ with the norm $\| g \|_\alpha = \| g \|_0 + v_\alpha(g)$.

Let $\varphi : J_\star \to \mathbb{R}$ be a function bounded from above and satisfying $v_\alpha(\varphi) < \infty$. For $y \in J_\star$ let

$$S_n\varphi(y) = \varphi(y) + \varphi(Ty) + \ldots + \varphi(T^{n-1}y)$$

Let $\mathcal{L}_\varphi(\Psi)(x) \stackrel{def}{=} \sum_{y \in T^{-1}x} e^{\varphi(y)}\ \Psi(y)$. Then $\mathcal{L}^n_\varphi(\Psi)(x) = \sum_{y \in T^{-n}x} e^{S_n(y)}\Psi(y)$

Lemma 4.5 If for some function φ as above the operator $\mathcal{L}_\varphi$ satisfies:

$$\| \mathcal{L}^n_\varphi \|_0 \le M \ \ \forall n \ge 1$$

Then:

$$\| \mathcal{L}^n_\varphi g \|_\alpha \le \ M e^{-\alpha n} \| g \|_\alpha + c \| g \|_0, \ \ \forall g \in \mathcal{H}_\alpha \ \ \square$$

Proof: Repeat word by word the proof from [DU]. Let $x,\ x' \in J_\star$ such that $\rho(x,x') = e^{-k}$, for some $k \geq 1$. By the definition of the metric, they are in the same k-cylinder. Choose $y \in T^{-n}x$ and let $y' \in T^{-n}x'$ be in the same (n+k)-cylinder with y. Then

$$|g(y) - g(y')| \leq \| g \|_\alpha e^{-\alpha(k+n)} \tag{4.5}$$

$$|e^{S_n\varphi(y)} - e^{S_n\varphi(y')}| = |\frac{e^{S_n\varphi(y)}}{e^{S_n\varphi(y')}} - 1| e^{S_n\varphi(y')} \leq$$

$$\leq c \| \varphi \|_\alpha e^{-\alpha k} e^{S_n\varphi(y')} \tag{4.6}$$

Thus

$$\mathcal{L}_\varphi^n g(x) - \mathcal{L}_\varphi^n g(x') = \sum_{y \in T^{-n}x} [e^{S_n\varphi(y)} g(y) - e^{S_n\varphi(y')} g(y')] =$$

$$= \sum_{y \in T^{-n}x} e^{S_n\varphi(y)} [g(y) - g(y')] + \sum_{y \in T^{-n}x} g(y') [e^{S_n\varphi(y)} - e^{S_n\varphi(y)}] =$$

The second sum is estimated by (4.6) as follows :

$$c \| \varphi \|_\alpha \| \mathcal{L}_\varphi^n g \|_0 \rho(x,x')^\alpha \leq cM \| \varphi \|_\alpha \| g \|_0 \rho(x,x')^\alpha$$

The first sum can be majorated by

$$e^{-\alpha n} \| g \|_\alpha \rho(x,x')^\alpha \| \mathcal{L}_\varphi^n 1 \|_0 \leq M e^{-\alpha n} \| g \|_\alpha \rho(x,x')^\alpha$$

Lemma 4.6 If the function ψ denotes $-\log Jac_\omega^T$, then the operator $\mathcal{L}_\psi$ satisfies:

$$\| \mathcal{L}_\psi^n g \|_0 \leq M \| g \|_0 \quad \square$$

Proof: The operator $\mathcal{L}_\psi$ being positive, it is enough to prove that $\| \mathcal{L}_\psi^n 1 \|_0 \leq M$.

Using the definition of $\mathcal{L}_\psi$, one gets:

$$\mathcal{L}_\psi^n 1(x) = \sum_{y \in T^{-n}x} \frac{1}{Jac_\omega^{T^n}(y)} \tag{4.7}$$

As in the proof of Lemma 4.1 one can majorize the right side of (4.7) by $\frac{B}{\min_{1 \leq i \leq s} \omega(f(R_i \bigcap J_\star))} = M < \infty$.

Lemma 4.7 The image under $\mathcal{L}_\psi$ of any bounded subset of $\mathcal{H}_\alpha$ is relatively compact in $\mathcal{H}_0$. $\square$

Proof: Before actually proving this lemma we would like to point out that bounded sequences in $\mathcal{H}_\alpha$ may not have any convergent subsequence in $\mathcal{H}_0$ (Arzela- Ascoli's theorem does not apply because the space $(J_\star, \rho)$ is not compact). Consider for example the sequence χ_{C_n}; it is bounded in $\mathcal{H}_\alpha$ and has no convergent subsequence in the supremum norm. However, after applying $\mathcal{L}_\psi$ the values of χ_{C_n} are "redistributed" over $J_\star$ and multiplied with weights e^ψ that decrease to zero as the cylinders C_k get closer to the parabolic point.

Let $(g_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_\alpha$, $\| g_n \|_\alpha \leq M$. We want to show that one can find a function $g \in \mathcal{H}_0$ and a subsequence g_{n_k} such that $\| \mathcal{L}_\psi g_{n_k} - g \|_0 \to 0$.

By Lemma 4.6, the functions $\mathcal{L}_\psi g_n$ are uniformly bounded in $\mathcal{H}_0$ by some constant M_1.

For $x, x' \in J_\star$ we have:

$$|\mathcal{L}_\psi g_n(x) - \mathcal{L}_\psi g_n(x')| = |\sum_{y \in T^{-1}x} [e^{\psi(y)} g_n(y) - e^{\psi(y')} g_n(y')]| \leq$$

$$\leq \sum_{y \in T^{-1}x} |e^{\psi(y)} - e^{\psi(y')}||g_n(y)| + \sum_{y \in T^{-1}x} e^{\psi(y')} |g_n(y) - g_n(y')| \leq$$

$$\leq \| \mathcal{L}_\psi g_n \|_0 \rho(x,x')^\alpha v_\alpha(\psi) + \rho(x,x')^\alpha v_\alpha(g_n) \| \mathcal{L}_\psi 1 \|_0 \leq C\rho(x,x')^\alpha$$

Therefore the family $\mathcal{L}_\psi g_n$ are equicontinuous on $J_\star$. By Arzela- Ascoli theorem for the sets $\bigcup_{i=1}^k C_i$, we can extract subsequences $(g_n^k)_{n>0}$ such that for each k and for some continuous function g

$$\sup_{x \in \bigcup_{i=1}^k C_i} |\mathcal{L}_\psi g_n^k - g| \to 0 \ \text{ as } n \to \infty$$

Consider the diagonal subsequence $(g_n^n)_n$ and relabel it g_n. We get that for each k

$$\epsilon_n^k \stackrel{def}{=} \sup_{x \in \bigcup_{i=1}^k C_i} |\mathcal{L}_\psi g_n - g| \to 0 \ \text{ as } n \to \infty \tag{4.8}$$

Finally, to show that the subsequence $\mathcal{L}_\psi g_n$ is uniformly Cauchy, let $\epsilon > 0$ be fixed.

$$|\mathcal{L}_\psi g_n(x) - \mathcal{L}_\psi g_{n+p}(x)| \leq \sum_{y \in T^{-1}x} e^{\psi(y)} |g_n(y) - g_{n+p}(y)| \leq$$

$$\sum_{y \in T^{-1}x \bigcap (\bigcup_{i=1}^k C_i)} e^{\psi(y)} |g_n(y) - g_{n+p}(y)| + \sum_{y \in T^{-1}x \bigcap (\bigcup_{i=k+1}^\infty C_i)} e^{\psi(y)} |g_n(y) - g_{n+p}(y)| \leq$$

$$\leq \epsilon_n^k \| \mathcal{L}_\psi 1 \|_0 + (\| g_n \|_0 + \| g_{n+p} \|_0) \sup_{x \in J_\star} \sum_{y \in T^{-1}x \bigcap (\bigcup_{i=k+1}^\infty C_i)} e^{\psi(y)}$$

If we denote by ϵ^k the supremum in previous inequality, by an argument similar to the one in (4.1), we have that $\epsilon^k \to 0$ as $k \to \infty$, so for some large $k = k(\epsilon)$, $\epsilon^k \le \epsilon/2$. For this fixed k there exists some index n_0 such that $\epsilon_n^k \le \epsilon/2$ $\forall n \ge n_0$. Therefore $\| \mathcal{L}_\psi g_n - \mathcal{L}_\psi g_{n+p} \|_0 \le \epsilon$ $\forall n \ge n_0 = n_0(\epsilon)$

Lemmas 4.5, 4.6, 4.7 allow us to use the result of Ionescu Tulcea and Marinescu [ITM] which states that $\mathcal{L}_\psi$ on the complexification of $\mathcal{H}_0$ can be decomposed into the sum $\mathcal{L}_\psi = P + S$ of a projection P onto the finite dimensional space of eigenvectors with eigenvalues on the unit circle, and a contraction S of $\mathcal{H}_\alpha$.

On the other hand, Schweiger formalism applies for the system $(T, J_\star, \omega, \mathcal{R}_\star)$. In particular there exists a unique T -invariant probability equivalent to ω, call it $d\mu = \rho d\omega$. This measure is ergodic, $\log \rho \in L^\infty(\omega)$, and the following holds:

$$\int_{J_\star} g\rho d\omega = \int_{J_\star} g \circ T^n \rho d\omega = \int_{J_\star} g \mathcal{L}_\psi^n \rho d\omega$$

This computation shows that ρ is an eigenvector of $\mathcal{L}$ with eigenvalue 1 and that there are no other eigenvalues of the form $\lambda = e^{2\pi \frac{p}{q}}$ except for $\lambda = 1$. (Because if ρ_λ were an eigenvector for such a λ, then $\rho_\lambda d\omega$ would be a T^q-invariant measure, and by uniqueness in Schweiger formalism this means that $\rho_\lambda = \rho$.)

The finiteness part in the theorem of Ionescu Tulcea and Marinescu shows that it is impossible to have eigenvalues of modulus 1 whose arguments are not rational multiples of 2π.

Theorem 4.8 Let $\psi = -\log Jac_\omega^T$ for the harmonic measure on parabolic Julia set. Then

1. The operator $\mathcal{L}_\psi : L^\infty(\omega) \to L^\infty(\omega)$ has only one eigenvalue of absolute value 1, this eigenvalue is 1, and the eigenvector ρ is continuous. The eigenvector ρ is normalized such that $\int_{J_\star} \rho d\omega = 1$.

2. $\mathcal{L}_\psi = P + S$ where P is the projection $P(g) = (\int_{J_\star} g d\omega)\rho$

3. The operator S acts on $\mathcal{H}_\alpha$ and $\| S^n \|_\alpha \le c_1 \gamma_1^n$ for some $\gamma_1 \in (0,1)$. $\square$

In the sections that follow we will denote by ν the T-invariant measure equivalent to ω and by ρ its density with respect to ω. We will denote by μ the T-invariant measure equivalent to the δ-conformal measure m and r will be the density $\frac{d\mu}{dm}$.

Both invariant measures being ergodic by Schweiger's formalism, there are only two possible situations: either $\nu = \mu$ or $\nu \perp \mu$.

If $\nu = \mu$ then their Jacobians (with respect to T) are equal so

$$\log J_\omega^T + \rho - \rho \circ T = \log J_m^T + r - r \circ T$$

Thus

$$\log J^T_\omega - \delta \log |T'| = \gamma \circ T - \gamma \tag{4.9}$$

where γ is a bounded function. We use an observation from [DU3] that the homology equation (4.9) for T implies a similar homology equation for f.

Recall that $T(x) = f^{N(x)}(x)$ for some $N(x) \geq 2$ then $N(x)$ is constant on cylinders and

$$\log J^T_\omega = \sum_{i=0}^{N(x)-1} \log J^f_\omega(f^i(x))$$

$$\delta \log |T'(x)| = \delta \sum_{i=0}^{N(x)-1} |f'(f^i x)|$$

Therefore if we denote by $\Theta(x) = \log J^T_\omega(x) - \delta \log |T'(x)|$ we get that

$$\Theta(x) - \Theta(fx) = \log J^f_\omega(x) - \delta \log |f'(x)| \tag{4.10}$$

On the other hand (4.9) for T implies that

$$\Theta(x) = \gamma(f^{N(x)}(x)) - \gamma(x), \;\; \Theta(fx) = \gamma(f^{N(x)}(x)) - \gamma(fx) \tag{4.11}$$

Combining (4.10) and (4.11) we get

$$\log J^f_\omega(x) - \delta \log |f'(x)| = \gamma(fx) - \gamma(x) \;\; a.e. \; x \in J \tag{4.12}$$

Since the left side of the equation (4.12) is continuous and the tree of preimages $\bigcup_{n \geq 0} f^{-n}(x)$ of every point is dense in J , one can follow the proof in [PUZ] to show that the the function γ has a continuous representative and that the equality (4.12) holds everywhere in J.

We will prove that this leads to contradiction and therefore only $\omega \perp m$ can happen.

5 Thermodynamical Formalism for Countable State Systems

5.1 Entropy

In this section we prove that if $\omega \perp m$ then $Hdim(\omega) < Hdim(J_f)$.

Let μ be a T invariant probability measure on $J_\star$. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_k \ldots\}$ be a finite or countable partition of $J_\star$. Denote by $\underline{Q} = Q_{i_1} \ldots, Q_{i_n}$ the set

$Q_{i_1} \bigcap T^{-1}Q_{i_2} \bigcap \cdots \bigcap T^{-n+1}Q_{i_n}$ and by $\mathcal{W}_n(\mathcal{Q})$ or $\mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \ldots T^{-n+1}\mathcal{Q}$ the set of all such words.

Definition 5.1 Let $H_\mu(\mathcal{Q}) \overset{def}{=} \Sigma_n \mu(Q_k) \log \frac{1}{\mu(Q_k)}$.

Lemma 5.2 $H_\mu(\mathcal{Q} \vee \tilde{\mathcal{Q}}) \leq H_\mu(\mathcal{Q}) + H_\mu(\tilde{\mathcal{Q}})$.

Proof: See Lemma 1.17 in [Bo], page 28.

Lemma 5.3 If $\mathcal{Q}$ is a finite partition of $J_\star$ then there exists the limit

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \cdots \vee T^{-n+1}\mathcal{Q}) \overset{def}{=} H_\mu(T, \mathcal{Q}). \ \square$$

Proof: See Lemma 1.18 in [Bo], page 28.

Lemma 5.4 If $\mathcal{Q}$ is a finite partition, then for any k,

$$H_\mu(T, \mathcal{Q} \vee T^{-1}\mathcal{Q} \vee \cdots \vee T^{-k+1}\mathcal{Q}) = H_\mu(T, \mathcal{Q}). \ \square$$

Proof: See Lemma 2.2 in [Bo], page 46.

Definition 5.5 The entropy

$$h_\mu(T) \overset{def}{=} \sup H_\mu(T, \mathcal{Q}),$$

where the supremum is taken over all finite partitions $\mathcal{Q}$ of $J_\star$.

Note that the entropy may be equal to zero or infinity.

Lemma 5.6 Let $\mathcal{B} = \{B_1 \ldots, B_m\}$ be a finite partition of $J_\star$ and let $\mathcal{D}_n$ be a sequence of partitions with $\mathrm{diam}(\mathcal{D}_n) \to 0$ as $n \to \infty$.

Then there exists a sequence of partitions $\mathcal{E}_n$ of cardinal m, $\mathcal{E}_n = \{E_1^n, \ldots, E_m^n\}$ such that each E_i^n is the union of some elements of $\mathcal{D}_n$ and $\lim_{n\to\infty} \mu(E_i^n \Delta B_i) = 0$ for each $i = \overline{1, m}$. $\square$

Proof: As in Lemma 1.23 from [Bo], page 33, except that the partitions $\mathcal{D}_n$ may be countable.

Lemma 5.7 Let $\epsilon > 0$ and let $\mathcal{B}$ be a finite partition of $J_\star$.

Then there exists $\delta_0 > 0$ such that for all partitions $\mathcal{D}$ with $\mathrm{diam}(\mathcal{D}) < \delta_0$ the following holds :

$$H_\mu(\mathcal{B} \vee \mathcal{D}) - H_\mu(\mathcal{D}) \leq \epsilon \ \square$$

Proof: As in Lemma 2.3 from [Bo], page 47.

Lemma 5.8 Suppose $\mathcal{D}_n$ is a sequence of finite partitions with $\mathrm{diam}(\mathcal{D}_n) \to 0$. Then

$$h_\mu(T) = \lim_{n\to\infty} H_\mu(T, \mathcal{D}_n) \ \Box$$

Proof: See Proposition 2.4 in [Bo], page 48.

Theorem 5.9 Let μ be a T-invariant probability measure on $J_\star$ such that

$$\sum_i \mu(C_i) \log \frac{1}{\mu(C_i)} < \infty,$$

where C_i are the standard cylinders from the end of section 3. Then the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C})$$

and it is equal to $h_\mu(T)$. $\Box$

Proof: By the T-invariance of μ , the sums $\Sigma_i \mu(T^{-k}C_i) \log \frac{1}{\mu(T^{-k}C_i)}$ are finite for each k, and the sequence $\frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C})$ has a finite limit which we will denote by L.

To show that $h_\mu(T) \le L$, let $\mathcal{D}_n^0 = \{C_1, C_2, \ldots, C_n, \bigcup_{k\ge n+1} C_k\}$, and $\mathcal{D}_n = \mathcal{D}_n^0 \vee \cdots \vee T^{-n+1}\mathcal{D}_n^0$.

Then $\mathrm{diam}\mathcal{D}_n \to 0$ and

$$\lim_{n\to\infty} h_\mu(T, \mathcal{D}_n^0) = \lim_{n\to\infty} h_\mu(T, \mathcal{D}_n) = h_\mu(T). \tag{5.1}$$

Together with $H_\mu(T, \mathcal{D}_n^0 \vee \cdots \vee T^{-k+1}\mathcal{D}_n^0) \le H_\mu(T, \mathcal{C} \vee \cdots \vee T^{-k+1}\mathcal{C})$, it implies that $h_\mu(T, \mathcal{D}_n^0) \le L$ and therefore that $h_\mu(T) \le L$.

For the reversed inequality we will show that for any $\epsilon > 0$, $h_\mu(T) \ge L - \epsilon$.

Claim that if two partitions of $J_\star$, $\mathcal{B} = \{B_1, \ldots B_m, \ldots\}$ and $\mathcal{F} = \{B_1, \ldots B_m, J_\star \setminus \bigcup_{k=1}^m B_k\}$ satisfy

$$H_\mu(\mathcal{B}) \le H_\mu(\mathcal{F}) + \epsilon \tag{5.2}$$

then for an arbitrary partition $\mathcal{A}$ of $J_\star$ the following holds:

$$H_\mu(\mathcal{B} \vee \mathcal{A}) \le H_\mu(\mathcal{F} \vee \mathcal{A}) + \epsilon.$$

To prove the claim let $B = J_\star \setminus \bigcup_{k=1}^m B_k$. By (5.2) we get

$$\sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{1}{\mu(B_i)} - \mu(B) \log \frac{1}{\mu(B)} \le \epsilon$$

and

$$\sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{\mu(B)}{\mu(B_i)} \ \leq \epsilon.$$

$$H_\mu(\mathcal{B} \vee \mathcal{A}) \ - \ H_\mu(\mathcal{F} \vee \mathcal{A}) \ = \sum_{i=m+1}^{\infty} \ \sum_a \ \mu(B_i \bigcap A_a) \log \frac{\mu(B \bigcap A_a)}{\mu(B_i \bigcap A_a)} \ =$$

$$= \sum_{i=m+1}^{\infty} \ \mu(B_i) \Sigma_a \ \frac{\mu(B_i \bigcap A_a)}{\mu(B_i)} \log \frac{\mu(B \bigcap A_a)}{\mu(B_i \bigcap A_a)}$$

By the concavity of the function $x \mapsto \log x$ we get :

$$H_\mu(\mathcal{B} \vee \mathcal{A}) \ - \ H_\mu(\mathcal{F} \vee \mathcal{A}) \ \leq \sum_{i=m+1}^{\infty} \ \mu(B_i) \log (\sum_a \frac{\mu(B \bigcap A_a)}{\mu(B_i)}) \ =$$

$$= \sum_{i=m+1}^{\infty} \mu(B_i) \log \frac{\mu(B)}{\mu(B_i)} \ \leq \epsilon.$$

Therefore the claim is true.

Fix some $\epsilon > 0$. Then for all large enough n, the partitions $\mathcal{D}_n^0$ are fine enough to ensure that

$$H_\mu(\mathcal{C}) \leq H_\mu(\mathcal{D}_n^0) + \epsilon \quad \text{for all } n \geq n_\epsilon.$$

Therefore $\quad H_\mu(T^{-k}\mathcal{C}) \leq H_\mu(T^{-k}\mathcal{D}_n^0) + \epsilon$. Applying the claim k times we get :

$$H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-k}\mathcal{C}) \ \leq H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-k}\mathcal{C}) \ + \epsilon \leq$$

$$H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \cdots \vee T^{-k}\mathcal{C}) \ + \ 2\epsilon \leq$$

$$\leq H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \cdots \vee T^{-k}\mathcal{D}_n^0) \ + \ k\epsilon$$

$$\Rightarrow \ \frac{1}{k} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-k}\mathcal{C}) \ \leq \ \frac{1}{k} H_\mu(\mathcal{D}_n^0 \vee T^{-1}\mathcal{D}_n^0 \vee \cdots \vee T^{-k}\mathcal{D}_n^0) \ + \ \epsilon$$

$$\Rightarrow L \ \leq h_\mu(\mathcal{D}_n^0) \ + \ \epsilon = h_\mu(\mathcal{D}_n) \ + \epsilon$$

$$\Rightarrow L \leq h_\mu(T) + \epsilon. \qquad \square$$

5.2 Pressure

We will define the pressure for potentials ϕ in the space of functions

$$\mathcal{H}_\alpha^{'} \overset{def}{=} \{\phi : J_\star \to \mathbb{R}\,;\phi \text{ is bounded from above, and } v_\alpha(\phi) < \infty\,\}.$$

For such functions ϕ define :

$$S_n\phi(\underline{A}) = \sup_{x\in\underline{A}} \sum_{k=0}^{n-1} \phi(T^k x) \text{ for a } n-\text{word } \underline{A} \in \mathcal{W}_n(\mathcal{C}) \tag{5.3}$$

and

$$Z_n(\phi) = \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} \exp S_n\phi(\underline{A}) \tag{5.4}$$

Remark 5.10 For functions $\phi \in \mathcal{H}_\alpha^{'}$ the sequence $\frac{1}{n}\log Z_n(\phi)$ is decreasing . Define the pressure to be

$$P(\phi, \mathcal{C}) \overset{def}{=} P(\phi) \overset{def}{=} \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi)\ .$$

It can be a real number or it can equal $-\infty$.

Theorem 5.11 Let $\phi \in \mathcal{H}_\alpha^{'}$ and let μ be a T-invariant probability measure on $J_\star$ satisfying $H_\mu(\mathcal{C}) < \infty$. Then

$$h_\mu(T) + \int_{J_\star} \phi\, d\mu \ \leq\ P(\phi) \quad \square \tag{5.5}$$

Proof: If $\int \phi d\mu = -\infty$ there is nothing to prove. Assume that $\int \phi d\mu$ is finite. We will show that

$$\frac{1}{n} \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} (\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \ +\ \int_{\underline{A}} S_n\phi d\mu) \ =$$

$$= \frac{1}{n} (\sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} \mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \ +\ \int_{J_\star} S_n\phi d\mu\ ) \ =$$

$$= \frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C}) + \int_{J_\star} \phi d\mu \ \leq P(\phi) \tag{5.6}$$

Let $z_{\underline{A}} \in \underline{A}$ satisfy : $\int_{\underline{A}} S_n\phi d\mu) \ \leq \mu(\underline{A}) S_n\phi(z_A)$. Then

$$\frac{1}{n} \sum_{\underline{A}\in\mathcal{W}_n(\mathcal{C})} (\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} \ +\ \int_{\underline{A}} S_n\phi d\mu) \ \leq$$

$$\leq \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \mu(\underline{A}) \log \frac{\exp S_n\phi(z_A)}{\mu(\underline{A})} \leq$$

$$\leq \frac{1}{n} \log \Big(\sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \exp S_n\phi(z_A) \Big) \leq$$

$$\leq \frac{1}{n} \log \Big(\sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \sup_{\underline{A}} \exp S_n\phi(z) \Big) = \frac{1}{n} Z_n(\phi) \tag{5.7}$$

$$\Rightarrow \quad h_\mu(T) + \int \phi \, d\mu \leq P(\phi)$$

Let $\delta = Hdim(J)$.

Theorem 5.12 The function $\phi = -\delta \log |T'|$ has $P(\phi) = 0$. $\square$

Proof: Let $\underline{A} \in \mathcal{W}_m(\mathcal{C})$. The map $T^m : \underline{A} \to J_\star$ being one-to-one and onto, there is a natural identification between $\mathcal{W}_m(\mathcal{C})$ and the set $T^{-m}x$, given by $\underline{A} \in \mathcal{W}_m(\mathcal{C}) \longleftrightarrow x_{\underline{A}} = \underline{A} \bigcap T^{-m} J_\star$ for arbitrary points x in $J_\star$.

Fix such an x and let $x_{\underline{A}} = \underline{A} \bigcap T^{-m} J_\star$.

$$S_m\phi(\underline{A}) = \sup_{y \in \underline{A}} \sum_{i=0}^{m-1} \phi(T^i y)$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \sum_{i=0}^{m-1} \sup_{x,y \in \underline{A}} |\phi(T^i x) - \phi(T^i y)| \leq$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \| \phi \|_\alpha \sum_{i=0}^{m-1} e^{-(m-i)\beta} \leq$$

$$\leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \text{const} \tag{5.8}$$

Let Ψ be the fixed point of the Perron Frobenius operator, $\mathcal{L}_\phi \Psi = \Psi$. According to [DU3], it is bounded away from zero and infinity. By the inequality (5.8) we get :

$$S_m\phi(\underline{A}) \leq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}) + \text{const} + \log \Psi(x_{\underline{A}}) \quad \Rightarrow$$

$$\exp S_m\phi(\underline{A}) \leq c \exp(\sum_{x_{\underline{A}}} \phi(T^i x_{\underline{A}})) \Psi(x_{\underline{A}}) \quad \Rightarrow$$

$$Z_m(\phi) = \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \exp S_m \phi(\underline{A}) \leq$$

$$\leq c \sum_{x_{\underline{A}} \in T^{-m}x} \exp S_m \phi(x_{\underline{A}}) \Psi(x_{\underline{A}}) = c(\mathcal{L}^m \Psi)(x) = c\Psi(x).$$

The function Ψ being bounded (see Theorem 4.2), we get : $Z_m(\phi) \leq M_2$ and $P(\phi) = \lim_{m\to\infty} \frac{1}{m} Z_m(\phi) \leq 0.$

The opposite inequality follows identically, using

$$\sup_{y \in \underline{A}} \sum_{i=0}^{m-1} \phi(T^i y) \geq \sum_{i=0}^{m-1} \phi(T^i x_{\underline{A}}).$$

Theorem 5.13 Let $\phi = -\delta \log |T'|$. The probability measure μ equivalent to the δ-conformal measure is the unique T-invariant measure that satisfies:

$$H_\lambda(T) + \int_{J_\star} \phi \, d\lambda = 0 \quad \text{and} \quad H_\lambda(\mathcal{C}) < \infty. \quad \square \tag{5.9}$$

Proof: Let λ be a T-invariant measure that satisfies (5.9) First let us assume that $\mu \perp \lambda$. Then there exists a Borel subset of $J_\star$, call it B, satisfying

$$\mu(B) = 0, \ \lambda(B) = 1, \quad \text{and} \quad T^{-1}B = B \tag{5.10}$$

By Lemma 5.6 for the partitions $\mathcal{B} = \{B, J_\star \setminus B\}$ and $\mathcal{D}_m = \mathcal{D}_m^0 \vee T^{-1}\mathcal{D}_m^0 \vee \cdots \vee T^{-m+1}\mathcal{D}_m^0$ where $\mathcal{D}_m^0 = \{C_1, \ldots C_m, \bigcup_{k=m+1}^\infty C_k\}$, we can find sets F_m that are union of standard cylinders and that satisfy

$$(\mu + \lambda)(B \Delta F_m) \to 0 \ \text{ as } \ m \to \infty \tag{5.11}$$

By Theorems 5.11 and 5.12 we get:

$$0 = P(\phi) = \inf \frac{1}{n} H_\lambda(\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C}) + \int_B \phi d\,\lambda \leq \tag{5.12}$$

$$\leq \frac{1}{n}(H_\lambda(\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C}) + \int_B S_n \phi d\,\lambda) \Rightarrow$$

$$0 \leq \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} (\lambda(\underline{A}) \log \frac{1}{\lambda(\underline{A})} + \int_{\underline{A} \bigcap B} S_n \phi \, d\lambda)$$

By an argument similar to the one in Theorem 5.11, if $x \in J_\star$ is fixed and $x_{\underline{A}} = \underline{A} \bigcap T^{-n}x$, then

$$\int_B S_n \phi d\lambda \leq \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \lambda(\underline{A})[\ S_n \phi(x_{\underline{A}}) + \ \| \phi \|_\alpha] \ \leq$$

$$\leq \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} \lambda(\underline{A})\ S_n\phi(x_{\underline{A}})\ + c_1\ \Rightarrow$$

$$-c_1{+}1 \leq \sum_{\underline{A} \subset F_n} \lambda(\underline{A})\ [\ \log \frac{1}{\lambda(\underline{A})}\ \ \exp S_n\phi(x_{\underline{A}})\ ] + \sum_{\underline{A} \bigcap F_n = \emptyset} \lambda(\underline{A})\ [\ \log \frac{1}{\lambda(\underline{A})} + S_n\phi(x_{\underline{A}})\ ]\ \leq$$

$$\leq \lambda(F_n) \log \frac{\sum_{\underline{A} \subset F_n} \exp S_n\phi(x_{\underline{A}})}{\lambda(F_n)}\ +\ \lambda(J_\star \setminus F_n) \log \frac{\sum_{\underline{A} \bigcap F_n = \emptyset} \exp S_n\phi(x_{\underline{A}})}{\lambda(J_\star \setminus F_n)} =$$

$$\leq \lambda(F_n) \log [\ \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} e^{S_n\phi(x_{\underline{A}})}\ \chi_{F_n}(x_{\underline{A}})\ ] +$$

$$+\ \lambda(J_\star \setminus F_n) \log [\ \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} e^{S_n\phi(x_{\underline{A}})}\ \chi_{J_\star \setminus F_n}(x_{\underline{A}})\ ]\ +\ 2\ \sup_{[0,1]} x \log \frac{1}{x} \tag{5.13}$$

By $\phi \in \mathcal{H}_\alpha^{'}$ and by the boundedness of $\log \Psi$ we get that

$$m(\underline{A}) \asymp e^{S_n(x_{\underline{A}})} \int_{\underline{A}} e^{-S_n\phi(z)}\ dm(z)\ =\ e^{S_n(x_{\underline{A}})} \int_{\underline{A}} |(T^n)'|^h\ dm$$

Therefore

$$e^{S_n\phi(x_{\underline{A}})}\ \asymp\ m(\underline{A})\ \asymp\ \mu(\underline{A}) \tag{5.14}$$

and (5.13) becomes:

$$-c_2\ \leq\ \lambda(F_n) \log [\ \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} e^{\mu(\underline{A})}\ \chi_{F_n}(x_{\underline{A}})\ ] +$$

$$+\ \lambda(J_\star \setminus F_n) \log [\ \sum_{\underline{A} \in \mathcal{W}_n(\mathcal{C})} e^{\mu(\underline{A})}\ \chi_{J_\star \setminus F_n}(x_{\underline{A}})\ ]$$

$$= \lambda(F_n) \log(\mu(F_n))\ +\ \lambda(J_\star \setminus F_n) \log(\mu(J_\star \setminus F_n)) \tag{5.15}$$

By $\mu(F_n) \to \mu(B) = 0$ and $\lambda(F_n) \to \lambda(B) = 1$ as $n \to \infty$ the inequality (5.15) becomes $-c_2 \leq -\infty$.

Therefore λ and μ can not be orthogonal. In particular this gives that in the class of T-invariant ergodic probabilities there is at most one that satisfies (5.9). This will be enough for our purposes.

To prove uniqueness for the general case, decompose the measure λ into $\lambda = \tilde{\lambda} + \tilde{\mu}$ where $\tilde{\lambda} \perp \mu$ and $\tilde{\mu} \ll \mu$ and follow the proof of Theorem 1.22, page 31 in [Bo].

In order to prove that the equality (5.9) actually holds for the measure μ it is enough to show that $h_\lambda(T) + \int \phi \, d\lambda \geq 0$.

Fix $n > 0$. We have:

$$\frac{1}{n} H_\mu(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C}) + \int_{J_\star} \phi d\mu =$$

$$= \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n} [\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} + \int_{\underline{A}} S_n \phi \, d\mu] \geq$$

$$\geq \frac{1}{n} \sum_{\underline{A} \in \mathcal{W}_n} [\mu(\underline{A}) \log \frac{1}{\mu(\underline{A})} + \mu(\underline{A}) \inf_{x \in \underline{A}} S_n \phi(x)] \tag{5.16}$$

Using one more time $\mu(\underline{A}) \asymp e^{S_n\phi(x_{\underline{A}})} = |(T^n)'(x_{\underline{A}})|^{-\delta}$ and the distortion properties of T it follows that

$$\log \frac{1}{\mu(\underline{A})} \geq \delta \sup_{x \in \underline{A}} (\sum_{i=0}^{n-1} \log |T'|(T^i x)) + c = -\inf S_n \phi(x) + c$$

and

$$h_\mu(T) + \int_{J_\star} \phi \, d\mu \geq 0$$

Theorem 5.14 Let λ be a T-invariant measure with positive entropy satisfying $H_\lambda(\mathcal{C}) < \infty$. Then

$$Hdim(\lambda) \leq \frac{h_\lambda(T)}{\int_{J_\star} \log |T'| \, d\lambda} \tag{5.17}$$

Proof: Let $\alpha > \frac{h_\lambda(T)}{\int_{J_\star} \log |T'| \, d\lambda}$. We want to show that $Hdim(\lambda) \leq \alpha$. Let $\delta_0 > 0$ satisfy

$$\delta_o < \frac{\alpha}{\alpha_0} - 1 \quad \text{where} \quad \alpha_0 \stackrel{def}{=} \frac{h_\lambda(T)}{\int_{J_\star} \log |T'| \, d\lambda} \tag{5.18}$$

Then there exists some n_δ such that for all $n > n_\delta$ the following holds:

$$\frac{1}{n} H_\lambda(\mathcal{C} \vee T^{-1}\mathcal{C} \vee \cdots \vee T^{-n+1}\mathcal{C}) \leq (1 + \delta_0) h_\lambda(T). \tag{5.19}$$

Let $\epsilon > 0$. Let $n > n_\epsilon$ be large such that $\operatorname{diam}\underline{A} \leq \epsilon$ for all $\underline{A} \in \mathcal{W}_n(\mathcal{C})$.

Let $\mathcal{F}_n$ be the family of all n-generation cylinders having positive λ mass and let

$$X = \bigcap_{n>0} \bigcup_{\underline{A} \in \mathcal{F}_n} \underline{A}.$$

Then $\lambda(X)=1$ and if

$$H_{\alpha,\epsilon}(X) \stackrel{def}{=} \inf\{\sum(\mathrm{diam}B)^\alpha;\ \text{the sets B cover X and } \mathrm{diam}(B)<\epsilon\,\}$$

we get :

$$H_{\alpha,\epsilon}(X) \le \sum_{\underline{A}\in\mathcal{F}_n} (\mathrm{diam}\underline{A})^\alpha \asymp \sum_{\underline{A}\in\mathcal{F}_n} \frac{1}{|((T^n)'(x_{\underline{A}})|^\alpha} =$$

$$= \sum_{\underline{A}\in\mathcal{F}_n} e^{-\alpha S_n(\log|T'|)(x_{\underline{A}})} = \sum_{\underline{A}\in\mathcal{F}_n} \lambda(\underline{A}) e^{-\alpha S_n(\log|T'|)(x_{\underline{A}})+\log 1/\lambda(\underline{A})} \le$$

$$\le \exp \sum_{\underline{A}\in\mathcal{F}_n} [-\alpha\lambda(\underline{A})S_n(\log|T'|(x_{\underline{A}}) + \lambda(\underline{A})\log 1/\lambda(\underline{A})\ ]$$

By Lemma 3.9

$$|S_n(\log|T'|)(x_{\underline{A}}) - S_n(\log|T'|(z))| = |\frac{(T^n)'(x_{\underline{A}})}{(T^n)'(z)}| \le c_3 \quad \text{for all } z\in\underline{A} \tag{5.20}$$

Therefore

$$H_{\alpha,\epsilon}(X) \le \exp(\sum_{\underline{A}\in\mathcal{F}_n} [-\alpha\int_{\underline{A}} S_n(\log|T'|)\,d\lambda \ +\ \lambda(\underline{A})\log\frac{1}{\lambda(\underline{A})}]\ )\ e^{c_3} \le$$

$$\le c_4\ (\exp[\frac{-\alpha}{n}\int_X S_n(\log|T'|)\,d\lambda \ +\ \frac{1}{n}\sum_{\underline{A}} \lambda(\underline{A})\log\frac{1}{\lambda(\underline{A})}])^n \le$$

$$\le c_4(\exp[-\alpha\int_X \log(|T'|)\,d\lambda \ +\ (1+\delta)h_\lambda]\ )^n$$

By (5.18)

$$-\alpha\int_X \log(|T'|)\,d\lambda \ +\ (1+\delta_0)h_\lambda \ = -\alpha\int_X \log(|T'|)\,d\lambda \ +\ (1+\delta_0)\alpha_0\int_X \log(|T'|)\,d\lambda$$

$$= -[\alpha-\alpha_0(1+\delta_0)]\int \log|T'|\,d\lambda$$

The jump transformation being expanding and the way δ_0 was selected imply that $-[\alpha-\alpha_0(1+\delta_0)]\int\log|T'|\,d\lambda$ is negative, say equal to $-\delta_1$. We get

$$H_{\alpha,\epsilon}(X) \le c_4\ e^{-n\delta_1} \quad \text{for all } n\ge n_\epsilon. \tag{5.21}$$

Therefore $Hdim(\lambda)\le\alpha$ for all $\alpha\ge\frac{h_\lambda}{\int\log|T'|\,d\lambda}$. This is enough to conclude that $Hdim(\lambda)\le\frac{h_\lambda}{\int\log|T'|\,d\lambda}$

Proposition 5.15 If μ is the T-invariant measure equivalent to the δ conformal measure m, then

$$\sum_{n>0} \mu(C_n) \log \frac{1}{\mu(C_n)} < \infty \quad \square$$

Proof: Without loss of generality we can assume that $\inf J = 0$ and that $\sup J = 1$. The measures m and μ being boundedly equivalent, it is enough to show that

$$\sum_{n>0} m(C_n) \log \frac{1}{m(C_n)} < \infty$$

Use the fact that $|T'|^\delta$ is the Jacobian of m to get that $\int_{C_n} |T'|^\delta dm = 1$. By

$$M_1^{-1} \leq |\frac{T'(x)}{T'(x')}| \leq M_1 \quad \text{for } x, x' \text{ in the same cylinder} \tag{5.22}$$

we get that $|T'(x)|^\delta \asymp \frac{1}{m(C_n)}$ for all $x \in C_n$.

On the other hand if the cylinder C_n is $C_n = [a_n, b_n] \bigcap J$, then

$$\int_{a_n}^{b_n} |T'(x)| dx = \int_{a_n}^{b_n} T'(x) dx = T(b_n) - T(a_n) = 1 - 0 = 1$$

so according to (5.22) we get $|T'(x)| \asymp \frac{1}{|C_n|}$ for $x \in C_n$. (The notation $|E|$ means the diameter of the set E.)

By construction $|C_n| = |f_1^{-K_n} I|$ for some interval I that is mapped by f univalently onto [0,1], where K_n is the integer part of $\frac{n}{deg(f)-1}$. Using the estimates from Appendix we get

$$|T'(x)| \asymp \frac{1}{|C_n|} \asymp \frac{1}{n^2} \quad \text{for } x \in C_n$$

so $m(C_n) \asymp \frac{1}{n^{2\delta}}$. The measure m being finite we get that $1 \asymp \sum_{n>0} \frac{1}{n^{2\delta}}$, which implies that $2\delta > 1$. Therefore

$$\sum_{n>0} m(C_n) \log \frac{1}{m(C_n)} \leq \text{ const } + \sum_{n>0} \frac{1}{n^{2\delta}} \log \frac{1}{n^{2\delta}} < \infty$$

for $2\delta > 1$.

Proposition 5.16 If ν is the T-invariant measure equivalent to the harmonic measure ω, then

$$\sum_{n>0} \nu(C_n) \log \frac{1}{\nu(C_n)} < \infty \quad \square$$

Proof: It is enough to show that $\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} < \infty$.

As in the previous proof we will use the fact that $|C_n| \asymp \frac{1}{n^2}$ and $|x| \asymp \frac{1}{n}$ for $x \in C_n$ to estimate the harmonic measure $\omega(C_n)$. Let $\inf C_n = a_n,\ \sup C_n = b_n,\ c_n = \frac{a_n+b_n}{2},\ r_n = \frac{b_n-a_n}{2}$

Let us consider the auxiliary domain Ω_n contained in $\mathbb{C} \setminus [0,\infty)$ whose boundary coincides with

$$\begin{aligned} \partial\Omega_n \bigcap B(c_n, r_n) &= J \bigcap B(c_n, r_n) \\ \partial\Omega_n \bigcap ([0,\infty) \setminus B(c_n, r_n)) &= [0,\infty) \setminus B(c_n, r_n) \end{aligned} \tag{5.23}$$

Let ω_{Ω_n} be the harmonic measure in the domain Ω_n. It is clear that

$$\omega(C_n, \cdot) \geq \omega_{\Omega_n}(C_n, \cdot) \tag{5.24}$$

The Julia set J being uniformly perfect gives

$$\omega_{\Omega_n}(J \bigcap B(c_n, r_n), y) \geq c(\beta) \quad \text{for } \forall y \in B(c_n, \frac{1}{2} r_n)$$

for some constant $c(\beta)$ depending only on the constant β from the definition of uniformly perfectness of J. In particular if w denotes the harmonic function in $\mathbb{C} \setminus [0,\infty)$ that vanishes on $[0,\infty) \setminus B(c_n, \frac{1}{2} r_n)$ and which equals 1 on $[p,\infty) \bigcap B(c_n, \frac{1}{2} r_n)$ we conclude that for every $y \in \mathbb{C}$ we have:

$$\omega_{\Omega_n}(J \bigcap B(c_n, r_n), y) \geq c(\beta)\ w(y) \tag{5.25}$$

The inequality (5.24) becomes

$$\omega(C_n, \infty) \geq \omega_{\Omega_n}(C_n, \infty) \geq c(\beta)\ \omega(\infty) \tag{5.26}$$

Using the change of coordinates $z \longrightarrow z^2$ the upper half plane is mapped onto the domain $\mathbb{C} \setminus [0,\infty)$ so one can easily estimate

$$w(\infty) \geq c r_n c_n^{-1/2} \asymp n^{-3/2}$$

Finally this computation together with (5.26) gives

$$\omega(C_n) \geq c n^{-3/2} \geq c' |x|^{3/2} \ \ \forall x \in C_n \tag{5.27}$$

which implies:

$$\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} \leq \sum_{n>0} \omega(C_n) \inf_{x \in C_n} \log \frac{1}{|x|^{3/2}} + c.$$

So

$$\sum_{n>0} \omega(C_n) \log \frac{1}{\omega(C_n)} \le c'' \int_J \log \frac{1}{|x|} d\omega(x) + c' = c'' \mathcal{U}_\omega(0) + c'$$

where $\mathcal{U}_\omega$ is the potential of the harmonic measure. The Julia set being uniformly perfect, it is regular for the Dirichlet problem; as a consequence the potential $\mathcal{U}_\omega$ is finite everywhere on J. In particular $\mathcal{U}_\omega(0) < \infty$.

Applying Theorems 5.11, 5.13, 5.14 we get that if the invariant harmonic measure and the invariant conformal measure are orthogonal, then the dimension of the harmonic measure is less than the dimension of the Julia set.

6 Solving the homology equation. Preliminaries

Throughout this section we use the notation $\frac{u(x)}{v(x)} = \lim_{z \to x} \frac{u(z)}{v(z)}$, $x \in J$, for positive harmonic functions u, v in $N \setminus J$, vanishing on J (here N is a neighborhood of J). The limit exists by the Main Lemma.

In this section we are going to prove that the homology equation (4.12) leads to a contradiction. First we are going to introduce some new notations: let Γ be a cone with the vertex at the parabolic point p, containing J, such that the forward orbit of the critical points does not intersect Γ. for $x \in J$ Let $D(x)$ denote the largest disc centered at x and contained in the cone Γ ,and let $\Gamma_x = \mathrm{conv}(p, D(x))$. Denote by f_1^{-1} the inverse branch of f that sends the parabolic point to itself.All the functions f_1^{-n} are well defined in Γ .

For a set E we will denote $f_1^{-n}E$ by E_{-n} .

The homology equation can be written as follows:

$$\log \frac{G(x)}{G(x_{-n})} - \delta \log |(f^n)'(x_{-n})| = \gamma(x) - \gamma(x_{-n}) \tag{6.1}$$

Lemma 6.1 If the homology equation (6.1) holds for some continuous γ, then $\forall z \in \Gamma \setminus J$

$$\exists \text{ the non-zero and finite limit } \lim_{n\to\infty} |(f^n)'(z_{-n})|^\delta \, G(z_{-n}). \quad \square$$

Proof: Let us estimate $\log(\frac{\beta_{n+k}(z)}{\beta_k(z)})$ where $\beta_n \overset{def}{=} |(f^n)'(z_{-n})|^\delta \, G(z_{-n})$.

Let x be a point of J closest to z that contains z in $D(x)$. Then:

$$\log \frac{\beta_{n+k}(z)}{\beta_k(z)} = \log \left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)} \right] \; + \; \log \frac{\beta_{n+k}(x)}{\beta_k(x)} = \tag{6.2}$$

$$= \log\left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)}\right] + \log\frac{G(x_{-n-k})}{G(x_{-k})}\,|(f^n)'(x_{-n-k})|^\delta =$$

$$= \log\left[\frac{\beta_{n+k}(z)}{\beta_k(z)} : \frac{\beta_{n+k}(x)}{\beta_k(x)}\right] + \gamma(x_{-n-k}) - \gamma(x_{-k})$$

The last term tends to zero (uniformly in n and z) when $k \to \infty$. The first term can be written as:

$$\log[\frac{G(z_{-n-k})}{G(z_{-k})} : \frac{G(x_{-n-k})}{G(x_{-k})}] - \delta\log|\frac{(f^k)'(z_{-k})}{(f^{n+k})'(z_{-n-k})} : \frac{(f^k)'(x_{-k})}{(f^{n+k})'(x_{-n-k})}|$$

Let us consider the first term of the previous sum; it can be written as :

$$\log[\frac{G(f_1^{-n}z_{-k})}{G(f_1^{-n}x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}]$$

or as

$$\log[\frac{(G\circ f_1^{-n})(z_{-k})}{(G\circ f_1^{-n})(x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}].$$

But $\frac{\mathrm{dist}(z_{-k},x_{-k})}{r(D(x_{-k}))} \asymp \frac{1}{k} \to 0$ as $k \to \infty$ and the Main Lemma applied to $u = G\circ f_1^{-n}$, $v = G$ in the domain $D(x_{-k})$ gives that

$$\log[\frac{(G\circ f_1^{-n})(z_{-k})}{(G\circ f_1^{-n})(x_{-k})} : \frac{G(z_{-k})}{G(x_{-k})}] \le c\frac{1}{k^\eta} \to 0$$

uniformly in n and z.

The second term can be written as

$$\delta\log|\frac{(f_1^{-n})'(x_{-k}}{(f_1^{-n})'(z_{-k})}| \le c\frac{1}{k} \to 0$$

uniformly in z and n as f_1^{-n} are univalent functions and $\frac{\mathrm{dist}(z_{-k},x_{-k})}{r(D(x_{-k}))} \asymp \frac{1}{k}$. This is enough to conclude that the limit exists.

Let us denote $v(z) = \lim_{n\to\infty} |(f^n)'(z_{-n})|^\delta\, G(z_{-n})$ for $x \in \Gamma$. As the limit in Lemma 6.1 exists, we conclude that

$$v(fz) = v(z)|f'(z)|^\delta \text{ if } z, f(z) \in \Gamma \tag{6.3}$$

The function v is not harmonic.

Let $\rho\Gamma$ denote a smaller cone with vertex at p.

Lemma 6.2 In the cone $\rho\Gamma$ the following holds:

$$c_\rho^{-1} \leq \frac{G(z)}{v(z)} \leq c_\rho$$

Proof: First we write for $z \in \rho D(x_{-m})$:

$$\frac{v(z)}{G(z)} = \lim_{n\to\infty}[\frac{G(f_1^{-n}z)}{G(z)} : \frac{G(f_1^{-n}x_{-m})}{G(x_{-m})}][\frac{G(f_1^{-n}x_{-m})}{G(x_{-m})}|(f^n)'(x_{-m-n})|^\delta]\cdot$$

$$\cdot \lim_{n\to\infty}[|\frac{(f^n)'(z_{-n})}{(f^n)'(x_{-n-m})}|^\delta] = \lim_{n\to\infty}[g_n^m(z)\Gamma_n^m] \lim_{n\to\infty}[\phi_n^m(z)]$$

Let us show that

$$\limsup_{n\to\infty} |\log g_n^m(z)| + |\log \Gamma_n^m| + |\log \phi_n^m(z)| \leq c < \infty$$

where c is independent of m and $z \in \rho D(x_{-m})$.

$$g_n^m(z) = \frac{(G\circ f_1^{-n})(z)}{G(z)} : \frac{(G\circ f_1^{-n})(x_{-m})}{G(x_{-m})} = \frac{u(z)}{v(z)} : \frac{u(x_{-m})}{v(x_{-m})}$$

and so it is bounded from above and bellow by constants which depend only on ρ and on the uniform perfectness of J. This follows from the Main Lemma.

$$|\log \Gamma_n^m| = \left| -\left[\log \frac{G(x_{-m})}{G(x_{-m-n})} - \delta \log |(f^n)'(x_{-m-n})|\right]\right| =$$

$$= \left| -[\log \frac{G(f^n x_{-n-m})}{G(x_{-m-n})} - \delta \log |(f^n)'(x_{-m-n})|]\right| =$$

$$= | - [\gamma(x_{-m}) - \gamma(x_{-m-n})]| \leq 2 \sup_{J_\star} |\gamma|$$

The last expression is

$$\phi_n^m(z) = |\frac{(f_1^{-n})'(x_{-m})}{(f_1^{-n})'(z)}|^\delta \asymp c'_\rho$$

by Koebe's theorem as f_1^{-n} is univalent in $D(x_{-m})$ and $z \in \rho D(x_{-m})$.

Lemma 6.3 For $x \in J \bigcap \Gamma$,

$$\frac{v(x)}{G(x)} = \lim_{\substack{z\to x\\ z\in\Gamma}} \frac{v(z)}{G(z)} = e^{-\gamma(x)} e^{\gamma(p)}$$

Proof: Fix $\rho > 0$. Let $z \to x$ in $\rho D(x_{-m}) \bigcap J$. Changing x_{-m} to x in all appearances of Lemma 6.2 we get that the limit exists and it is equal to $e^{-\gamma(x)} e^{\gamma(p)}$.

Notation We are going to denote $v_p \stackrel{def}{=} v$ and we will call it "parabolic point automorphic function". We will introduce another function of this type and show that they are proportional.

6.1 Repelling point automorphic function

We can choose x_0 to be a repelling periodic point of f and let it be close to p such that

$$\mathrm{orb}(\mathrm{Crit}(f)) \bigcap D(x_0, |x_0 - p|) = \emptyset.$$

Let l be the period of x_0 and let $(f^l)'(x_0) = \lambda > 1$.

Let $F = f^l$ and F^{-n} be the holomorphic inverse branches that send x_0 to x_0. These functions are single-valued holomorphic functions in the whole disc $D(x_0, |x_0 - p|)$.

Lemma 6.4 For all $z \in D(x_0, |x_0 - p|) \setminus J$,

$$\exists \text{ the non zero and finite limit } \lim_{n \to \infty} |(F^n)'(F^{-n} z)|^\delta G(F^{-n} z). \quad \Box$$

Proof: Let us estimate $\log[b_{n+k}(z) : b_k(z)]$ where $b_n \stackrel{def}{=} |(F^n)'(F^{-n} z)|^\delta G(F^{-n} z)$.

Let $x \in J$ be the point closest to z . Then $z \in D(x_0, |x_0 - p|)$ and

$$\log \frac{b_{n+k}(z)}{b_k(z)} = \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + \log \frac{b_{n+k}(x)}{b_k(x)} =$$

$$= \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + \log \frac{G(F^{-n-k} x)}{G(F^{-k} x)} |(F^n)'(F^{-n-k} x)|^\delta =$$

$$= \log[\frac{b_{n+k}(z)}{b_k(z)} : \frac{b_{n+k}(x)}{b_k(x)}] + (\gamma(F^{-n-k} x) - \gamma(F^{-k} x)).$$

The last term tends to zero when $k \to \infty$ uniformly in n and in $z \in \rho D(x_0, |x_0 - p|)$ for every $\rho < 1$.

The first term can be written as

$$\log[\frac{G(F^{-n-k} z)}{G(F^{-k} z)} : \frac{G(F^{-n-k} x)}{G(F^{-k} x)}] + \delta \log |\frac{(F^{n+k})'(F^{-n-k} z)}{(F^k)'(F^{-k} z)} : \frac{(F^{n+k})'(F^{-n-k} x)}{(F^k)'(F^{-k} x)}| \tag{6.4}$$

Let us rewrite it as:

$$\log[\frac{(G \circ F^{-n-k})(z)}{(G \circ F^{-k})(z)} : \frac{(G \circ F^{-n-k})(x)}{(G \circ F^{-k})(x)}] =$$

$$\log[\frac{(G\circ F^{-n})(F^{-k}z)}{G(F^{-k}z)} : \frac{(G\circ F^{-n})(F^{-k}x)}{G(F^{-k}x)}].$$

But if $z \in \rho D(x_0, |x_0 - p|)$ then

$$\frac{\mathrm{dist}(F^{-k}z, F^{-k}x)}{|x_0 - p|} \leq c_\rho \lambda^{-k} \tag{6.5}$$

Considering $u = G \circ F^{-n}$, $v = G$ in $D(x_0, |x_0 - p|)$, from the Holder continuity of $\log \frac{u}{v}$ we conclude that

$$|\log \frac{u(F^{-k}z)}{v(F^{-k}z)} : \frac{u(F^{-k}x)}{v(F^{-k}x)}| \leq c_1(\rho)\lambda^{-k\eta} \to 0 \text{ as } k \to \infty$$

uniformly in u, v, z when $z \in \rho D(x_0, |x_0 - p|)$ .That means that the first term of (6.4) tends to zero when $k \to \infty$ uniformly in n and $z \in \rho D(x_0, |x_0 - p|)$.

The second term of (6.4) can be written as

$$\delta \log |\frac{(F^n)'(F^{-n-k}z)}{(F^n)'(F^{-n-k}x)}| = \delta \log |\frac{(F^{-n})'(F^{-k}x)}{(F^{-n})'(F^{-k}z)}|$$

Now (6.5) and Koebe's principle applied to the univalent function F^{-n} in $D(x_0, |x_0 - p|)$ show that

$$\delta \log |\frac{(F^{-n})'(F^{-k}x)}{(F^{-n})'(F^{-k}z)}| \leq c_2(\rho)\lambda^{-k} \to 0 \text{ as } k \to \infty \tag{6.6}$$

where the convergence is uniform in n and $z \in \rho D(x_0, |x_0 - p|)$.

Let us denote $v_{x_0}(z) = \lim_{n\to\infty} |(F^n)'(F^{-n}z)|^\delta G(F^{-n}z)$, for $z \in D(x_0, |x_0 - p|)$. As the limit in the previous lemma exists we can conclude that

$$v_{x_0}(Fz) = v_{x_0}(z)|F'(z)|^\delta \text{ for } Fz \in D(x_0, |x_0 - p|). \tag{6.7}$$

The function v_{x_0} is not harmonic.

Lemma 6.5 In the disc $\rho D(x_0, |x_0 - p|)$,

$$c_3(\rho)^{-1} \leq \frac{G(z)}{v_{x_0}(z)} \leq c_3(\rho) \quad \square$$

Proof:

$$\frac{v_{x_0}(z)}{G(z)} = \lim_{n\to\infty}[\frac{G(F^{-n}z)}{G(z)} : \frac{G(F^{-n}x_0)}{G(x_0)}][\frac{G(F^{-n}x_0)}{G(x_0)}|(F^n)'(x_0)|^\delta]|\frac{(F^{-n})'(x_0)}{(F^{-n})'(z)}|^\delta \tag{6.8}$$

The middle factor equals 1 by homology equation. The last factor is bounded independently of n and $z \in \rho D(x_0, |x_0 - p|)$, by Koebe's principle. The first factor is bounded independently of n and $z \in \rho D(x_0, |x_0 - p|)$ by the Main Lemma for $u = G \circ F^{-n}$, $v = G$ in $D(x_0, |x_0 - p|)$.

Lemma 6.6 For $x \in J \bigcap D(x_0, |x_0 - p|)$

$$\frac{v_{x_0}(x)}{G(x)} = \lim_{z\to x, z\in D} \frac{v_{x_0}(z)}{G(z)} = e^{-\gamma(x)}e^{\gamma(x_0)} \quad \square$$

Proof: Let z approach $x \in \rho D(x_0, |x_0 - p|)$, for some $\rho < 1$.

$$\frac{v_{x_0}(z)}{G(z)} = \lim_{n\to\infty}[\frac{G(F^{-n}z)}{G(z)} : \frac{G(F^{-n}x)}{G(x)}][\frac{G(F^{-n}x)}{G(x)}|(F^n)'(F^{-n}x)|^\delta]|\frac{(F^{-n})'(x)}{(F^{-n})'(z)}|^\delta$$

The middle factor equals $e^{-\gamma(x)}e^{\gamma(F^{-n}x)}$ and thus tends to $e^{-\gamma(x)}e^{\gamma(x_0)}$ uniformly in $z \in \rho D(x_0, |x_0 - p|)$ because $x_{-n} \in \rho D(x_0, |x_0 - p|)$ when $n \to \infty$.

The logarithms of the first and the last factor can be estimated by $c(\rho)|x - z|^\eta$ and by $c(\rho)|x - z|$ respectively, using the Main Lemma and Koebe's distortion properties independently of n.

6.2 Coming to contradiction when $Hdim(J) > 3/4$

Choose $z_0 \in \frac{1}{2}\Gamma_{x_0}$ sufficiently close to x_0.

$$v(z) = v(f^n z_{-n}) = |(f^n)'(z_{-n})|^\delta v(z_{-n}) = \frac{1}{|(f_1^{-n})'(z)|^\delta} v(z_{-n}) \asymp n^{2\delta} v(z_{-n})$$

Therefore by the estimates from the Appendix

$$v(z_{-n}) \asymp \frac{v(z)}{n^{2\delta}} \asymp \frac{1}{n^{2\delta}} \tag{6.9}$$

On the other hand let us prove the following apriori estimate for Green's function:

$$G(z_{-n}) \geq c\frac{1}{n^{3/2}}. \tag{6.10}$$

Clearly $c''_{x_0} n^{-2} \leq |z_{-n} - x_{-n}| \leq c'_{x_0} n^{-2}$.

Consider $B_n = B(x_{-n}, 2c'_{x_0}n^{-2})$. Then $z_{-n} \in B_n$ and by Lemma 1.4 we get

$$G(\xi) \le c\, G(z_{-n}) \quad \forall \xi \in B_n.$$

and so as usual

$$\omega(J \bigcap \frac{1}{2}B_n) \le c\, G(z_{-n}).$$

It is enough to prove the estimate

$$\omega(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2})) \ge \frac{c}{n^{3/2}} \tag{6.11}$$

Let us consider the auxiliary domain Ω_n contained in $\mathbb{C} \setminus J$ whose boundary coincides with J in $(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2})$ and it is equal to the half-line $[p, \infty)$ outside of $B(x_{-n}, 2c'_{x_0}n^{-2})$. Note that

$$\omega(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2})) \ge \omega_{\Omega_n}(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2})).$$

By cap $(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2})) \ge \beta$ cap $B(x_{-n}, c'_{x_0}n^{-2})$ we get:

$$\omega_{\Omega_n}(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2}), y) \ge c(\beta) \quad \text{for } \forall y \in B(x_{-n}, 2c'_{x_0}n^{-2}) \tag{6.12}$$

In particular if w denotes the harmonic function in $\mathbb{C} \setminus [p, \infty)$ that vanishes on $[p, \infty) \setminus \frac{1}{2}B(x_{-n}, c'_{x_0}n^{-2})$ and which equals 1 on $[p, \infty) \bigcap \frac{1}{2}B(x_{-n}, c'_{x_0}n^{-2})$ we conclude that for every $z \in \mathbb{C}$ we have:

$$\omega_{\Omega_n}(J \bigcap B(x_{-n}, 2c'_{x_0}n^{-2}), z) \ge c(\beta)\, w(z) \tag{6.13}$$

On the other hand

$$w(z) \asymp c(\beta)c'_{x_0}n^{-2}|x_{-n} - p|^{1/2} \asymp c_{\beta, x_0}n^{-3/2}$$

and (6.11) is proved.

By Lemma 6.2

$$G(z_{-n}) \le c_\rho v(z_{-n})$$

So for some constant c independent of n we get $n^{-3/2} \le c\, n^{-2\delta}$ which is impossible if the dimension δ is greater than $3/4$.

To get the contradiction in the case when $\delta \le 3/4$ seems much more difficult. The next section is about it.

7 Solving the homology equation. Simultaneous linearization

As in the previous section we will work in a fixed cone Γ containing the Julia set with the vertex at the parabolic point p, such that the forward orbit of the critical points does not intersect Γ. For $x \in J$ denote by $D(x)$ the largest disc centered at x and contained in the cone Γ.

Let us consider the following functions:

$$\beta_p(z) = \sum_{n \geq 1} [\log |f'(z_{-n})| - \log |f'(x_{-n})|]$$

where $z_{-n} = f_1^{-n} z$ and $x_{-n} = f_1^{-n} x_0$ for $z \in \Gamma$, and

$$\beta_{x_0}(z) = \sum_{n \geq 1} [\log |F'(F^{-n} z)| - \log |F'(x_0)|]$$

for $z \in D(x_0, |x_0 - p|)$.

These are symmetric harmonic functions in the domains Γ, $D(x_0, |x_0 - p|)$, respectively.

Consider

$$h_p(z) \stackrel{def}{=} \int_{x_0}^{z} e^{-(\beta_p(z) + i\tilde{\beta_p}(z))}\, dz \tag{7.1}$$

and

$$h_{x_0}(z) \stackrel{def}{=} \int_{x_0}^{z} e^{-(\beta_{x_0}(z) + i\tilde{\beta x_0}(z))}\, dz. \tag{7.2}$$

Due to the symmetry of $\beta_p(z)$ and $\beta_{x_0}(z)$ one can choose their complex conjugates such that h_p and h_{x_0} are real on the real axis.

Because

$$\beta_p(fz) - \beta_p(z) = \log |f'(z)| \quad \text{for } z, f(z) \in \Gamma$$

$$\beta_{x_o}(Fz) - \beta_{x_0}(z) = \log |F'(z)| - \log \lambda \quad \text{for } z, F(z) \in D(x_0, |x_0 - p|)$$

we have that:

$$|\frac{h_p'(fz)}{h_p'(z)}| = \frac{e^{-\beta_p(fz)}}{e^{-\beta_p(z)}} = \frac{1}{|f'(z)|} \quad \text{for } z, fz \in \Gamma \tag{7.3}$$

$$|\frac{h_{x_0}'(Fz)}{h_{x_0}'(z)}| = \frac{\lambda}{|F'(z)|} \quad \text{for } z, Fz \in D(x_0, |x_0 - p|) \tag{7.4}$$

Let us consider Koenig's function (or Fatou coordinates) Φ_{x_0}:

$$\Phi_{x_0}(z) = \lim_{n\to\infty} \frac{F^{-n}(z) - x_0}{(F^{-n})'(x_0)}$$

It is well defined in $D(x_0, |x_0 - p|)$. See [CG] for details.

We can see that $\Phi'_{x_0}(z) = h'_{x_0}(z)$ and that both Φ_{x_0} and h_{x_0} vanish at x_0. Therefore

$$\Phi_{x_0} \equiv h_{x_0}$$

Similarly, if

$$\Phi_p(z) = \lim_{n\to\infty} \frac{f_1^{-n}(z) - f_1^{-n}(x_0)}{(f_1^{-n})'(x_0)}$$

then $\Phi_p(x_0) = 0$, and $\Phi'_p(z) = h'_p(z)$. Therefore $\Phi_p \equiv h_p$.

In particular

$$h_{x_0} \circ F(z) = \lambda\, h_{x_0}(z) \quad \text{for } z, Fz \in D(x_0, |x_0 - p|), \quad \lambda > 1 \tag{7.5}$$

$$h_p \circ f(z) = h_p(z) + a \quad \text{for } z, fz \in \Gamma, \quad a > 0 \tag{7.6}$$

$$h_p(x_0) = h_{x_0}(x_0) = 0, \; h_p(p) = \infty \tag{7.7}$$

Let us notice also that

$$v_p(z) = e^{\delta\beta_p(z)} \lim_{n\to\infty} G(z_{-n})|(f^n)'(x_{-n})|^\delta \quad \text{for } z \in \Gamma$$

$$v_{x_0}(z) = e^{\delta\beta_{x_0}(z)} \lim_{n\to\infty} G(F^{-n}z)|(F^n)'(x_0)|^\delta \quad \text{for } z \in D(x_0, |x_0 - p|)$$

In particular the limits in the right part exist. Let us denote them by $\tilde{\tau}_p(z)$ for $z \in \Gamma$, $\tilde{\tau}_{x_0}(z)$ for $z \in D(x_0, |x_0 - p|)$. These are positive harmonic functions defined in $\Gamma \setminus J$ and $D(x_0, |x_0 - p|) \setminus J$, respectively, and subharmonic in Γ and $D(x_0, |x_0 - p|)$ respectively, and vanishing on J.

By Lemma 6.2 and Lemma 6.5, if $\rho < 1$, then:

$$\frac{\tilde{\tau}_p(z)}{G(z)} \asymp e^{-\delta\beta_p(z)} c_\rho \quad \text{for } z \in \rho\Gamma$$

$$\frac{\tilde{\tau}_{x_0}(z)}{G(z)} \asymp e^{-\delta\beta_{x_0}(z)} c_\rho \quad \text{for } z \in \rho D(x_0, |x_0 - p|)$$

Because of Lemmas 6.3 and 6.6

$$\frac{\tilde{\tau}_p(x)}{G(x)} = e^{-\delta\beta_p(x)} e^{-\gamma(x)} e^{\gamma(p)} \quad \text{for } x \in \rho\Gamma \bigcap J$$

$$\frac{\tilde{\tau}_{x_0}(x)}{G(x)} = e^{-\delta\beta_{x_0}(x)}e^{-\gamma(x)}e^{\gamma(x_0)} \quad \text{for } x \in \rho D(x_0, |x_0 - p|) \bigcap J$$

Denoting $\tau_p = \tilde{\tau}_p(z)e^{-\gamma(p)}$, $\tau_{x_0} = \tilde{\tau}_{x_0}(z)e^{-\gamma(x_0)}$ we conclude that

$$\frac{\tau_p(x)}{\tau_{x_0}(x)} = \left|\frac{h'_p(x)}{h'_{x_0}(x)}\right|^{\delta}$$

If we introduce

$$A(z) \stackrel{def}{=} (\frac{h'_p(z)}{h'_{x_0}(z)})^{\delta/2}$$

we see that A is a well defined holomorphic function in $\Gamma \bigcap D(x_0, |x_0 - p|)$ and

$$\frac{\tau_p(x)}{\tau_{x_0}(x)} = |A(x)|^2 \tag{7.8}$$

Lemma 7.1 (On harmonic rigidity)
Let u, v be two non-negative subharmonic functions in the disc B with diameter I. Let $J \subset I$ be a closed, uniformly perfect set with infinitely many components, $H_1(J) = 0$.
If u, v vanish on J and are positive and harmonic in $B \setminus J$, and if

$$\frac{u(x)}{v(x)} = \lim_{z \to x} \frac{u(z)}{v(z)} = |A(x)|^2 \,,\ \forall x \in J \tag{7.9}$$

for some holomorphic function A in B, then $|A| \equiv$ constant. $\Box$

Proof: Let us consider $\partial u(z)$ and $\partial v(z)$. These are holomorphic functions in $B \setminus J$ and their $\bar\partial$ derivatives are the Riesz measures μ_u, μ_v, respectively. Consider the following function:

$$w_1(z) = \partial u(z) - A(z)\overline{A(\bar z)}\partial v(z).$$

It is holomorphic in $B \setminus J$ and its distributional $\bar\partial$ derivative is equal to

$$\bar\partial\, w_1 \;=\; \mu_u - |A(x)|^2\mu_v \;\equiv\; 0$$

by the assumption (7.9) . So $w_1(z)$ is a holomorphic function in B. In a similar way one can show that $w_2 \stackrel{def}{=} \frac{\partial u}{\partial \bar z} - \overline{A(z)}A(\bar z)\frac{\partial v}{\partial \bar z}$ is anti-holomorphic in the ball B.
In particular $\partial u(x) - |A(x)|^2 \partial v(x)$ is continuous and locally bounded on I, the diameter of the ball B.

Without loss of generality we can assume that $u(\bar{z}) = u(z),\ v(\bar{z}) = v(z)$; otherwise we can consider $u^\star(z) = u(z) + u(\bar{z}),\ v^\star(z) = v(z) + v(\bar{z})$ and (7.9) still holds with symmetrized functions.

But if $u,\ v$ are symmetric, then on I we have $\partial u = \frac{\partial u}{\partial x},\ \partial v = \frac{\partial v}{\partial x}$ and we conclude that

$$\frac{\partial u}{\partial x} - |A(x)|^2 \frac{\partial v}{\partial x} \quad \text{is continuous on } I \tag{7.10}$$

Next we will prove that $W(x) \overset{def}{=} u(x) - |A(x)|^2 v(x)$ is locally C^1 on I.

By $\frac{\partial}{\partial x} = (\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})$ we get that for $a, b \in \mathbb{R}$ and $\epsilon > 0$

$$W(b + i\epsilon) - W(a + i\epsilon) = \int_a^b \frac{\partial W}{\partial z}(t + i\epsilon) + \frac{\partial W}{\partial \bar{z}}(t + i\epsilon) dt \tag{7.11}$$

By $W(z) = u(z) - A(z)\overline{A(\bar{z})}v(z)$ we get that

$$\frac{\partial W}{\partial z} = \frac{\partial u}{\partial z} - A'(z)\overline{A(\bar{z})}v(z) - A(z)\overline{A'(z)}v(z) - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial z}$$

and

$$\frac{\partial W}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial \bar{z}}$$

This implies that for $z \in B \setminus J$

$$\frac{\partial W}{\partial x}(z) = [\ \frac{\partial u}{\partial z} - A(z)\overline{A(\bar{z})}\frac{\partial v}{\partial z}\ ] - v(z)[\ A'(z)\overline{A(\bar{z})} + A(z)\overline{A'(z)}\ ] +$$

$$+ [\ \frac{\partial u}{\partial \bar{z}} - A(z)\overline{A(z)}\frac{\partial v}{\partial \bar{z}}\ ] + \frac{\partial v}{\partial \bar{z}}[\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)] = w_1 - w_3 + w_2 + w_4$$

The functions $w_1,\ w_2,\ w_3$ are continuous on a neighborhood of I, because they are either analytic or anti-analytic or the product between two continuous functions.

The function $w_4 = \frac{\partial v}{\partial \bar{z}}[\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)]$ is bounded uniformly in ϵ because

$$|\overline{A(z)}A(z) - \overline{A(\bar{z})}A(z)| \leq c\epsilon \quad \text{for Im } z = \epsilon$$

and

$$|\frac{\partial v}{\partial \bar{z}}| \leq \int_J \frac{d\mu_v(\zeta)}{|\zeta - z|} \leq \frac{c}{\epsilon} \quad \text{for Im } z = \epsilon$$

Equation (7.11) becomes:

$$|W(b + i\epsilon) - W(a + i\epsilon)| \leq \int_a^b |w_1(t + i\epsilon) - w_3(t + i\epsilon) + w_2(t + i\epsilon) + w_4(t + i\epsilon)| dt$$

Sending ϵ to zero one gets that $|W(b) - W(a)| \leq M|b-a|$, therefore W is absolutely continuous on $\mathbb{R}$ and

$$\frac{\partial W}{\partial x} = w_1(x) - w_3 + w_2 + 0$$

is correct for $x \in I \setminus J$, and therefore on I because $H_1(J) = 0$. Therefore W is in $C^1_{loc}(I)$.

Its derivative is equal to

$$[\frac{\partial u}{\partial x} - |A(x)|^2 \frac{\partial v}{\partial x}] - v(x)(|A(x)|^2)'$$

Consider the function W on one interval L of the complement of J in I.

It vanishes at the endpoints of L, so its derivative has a zero at a certain $x_l \in L$:

$$(\ u - |A|^2 v\ )'(x_l) \ = \ 0 \tag{7.12}$$

Because the function $u - |A|^2 v$ vanishes on J which is a perfect set we can conclude that its derivative vanishes on J also.

On the other hand on the real line

$$(u - |A|^2 v)' = \frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x} - v(|A|^2)' = w_1 - (|A|^2)' v$$

and both $(u - |A|^2 v)'$ and $(|A|^2)' v$ vanish on J, therefore $w_1(x) = \frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x}$ has to be zero on J. But this function is holomorphic in a ball containing J, so it has to vanish everywhere : $w_1(x) \equiv 0$ on I and

$$\frac{\partial u}{\partial x} - |A|^2 \frac{\partial v}{\partial x} \equiv 0 \quad \text{on } I. \tag{7.13}$$

Computing the the x-derivative in (7.12) we get that

$$\frac{\partial u}{\partial x}(x_l) - |A|^2(x_l) \frac{\partial v}{\partial x}(x_l) - v(x_l)(|A|^2)'(x_l) = 0$$

which according to (7.13) gives that $v(x_l)(|A|^2)'(x_l) = 0$. The function v being harmonic outside of J, can not vanish at x_l, which means that $(|A|^2)'(x_l) = 0$.

This is enough to conclude that $|A|^2 =$ constant . $\square$

Corollary 7.2 If u, v are symmetric then $u = \text{const } v$. $\square$

Proof: By previous lemma, $\Delta(u - cv) = 0$. So the function $u - cv$ is harmonic in the ball B. Its zero set, $Z(u - cv)$, is locally a real analytic curve, otherwise $u \equiv cv$.

Thus if $u \neq c\, v$ then at least $u = c\, v$ on the union of some intervals of the real line, which means that $\frac{\partial u}{\partial x} - c\,\frac{\partial v}{\partial x} = 0$ on some intervals on the real line. Due to the symmetry of $u,\ v$ the y-derivative of both functions vanishes on $\mathbb{R}$, so the following holds:

$$\partial u - c\,\partial v = 0 \quad \text{on some intervals of the real line}$$

therefore $\partial u - c\,\partial v = 0$ everywhere in B, because it is holomorphic.

Similarly $\bar\partial u - c\,\bar\partial v = 0$ everywhere in B. This implies that $u - c\, v$ is constant in B , and being zero on the real line, it has to be identically zero in B. Therefore $u \equiv c\, v$ $\square$

Applying Lemma 7.1 on harmonic rigidity (according to [A1] or [DU3] the Julia set has linear measure zero) and its corollary for the functions $\tau_p,\ \tau_{x_0}$ that satisfy (7.8) in $\Gamma \bigcap D(x_0, |x_0 - p|)$, we get that the two functions are proportional. On the real axis

$$\frac{h_p'(x)}{h_{x_0}'(x)} = |\frac{h_p'(x)}{h_{x_0}'(x)}| = \text{ const } = |\frac{h_p'(x_0)}{h_{x_0}'(x_0)}| = \frac{e^{-\beta_p(x_0)}}{e^{-\beta_p(x_0)}} = 1$$

so actually $\tau_p(z) = \tau_{x_0}(z) \stackrel{def}{=} \tau(z)$ for $z \in \Gamma \bigcap D(x_0, |x_0 - p|)$.

So $h_p'(z) = h_{x_0}'(z)$ and $h_p(z) \equiv h_{x_0}(z)$ in $\Gamma \bigcap D(x_0, |x_0 - p|)$ because $h_p(x_0) = h_{x_0}(x_0) = 0$.

Let $h(z) \stackrel{def}{=} h_p(z) = h_{x_0}(z)$ for $z \in \Gamma \bigcap D(x_0, |x_0 - p|)$. The functions h_p and h_{x_0} being defined as Fatou coordinates are univalent and the following conjugations (linearization) hold:

$$h(z) = \lambda h(F^{-1}z) \quad z, F^{-1}z \in D(x_0, |x_0 - p|)$$

$$h(z) = h(f_1^{-1}z) + a \quad a > 0, z, f_1^{-1}z \in \Gamma$$

$$h(x_0) = 0, h(p) = \infty \tag{7.14}$$

Consider the domain $\mathcal{O} = h(\Gamma \bigcap D(x_0, |x_0 - p|))$ and let $R(w) = \tau(h^{-1}w),\ w \in \mathcal{O}$.

The map R is defined on $\mathcal{O}$ and

$$R(\lambda z) = \lambda^\delta R(z) \ \text{ for } \ z, \lambda z \in \mathcal{O}$$

$$R(z + a) = R(z) \quad \text{for} \ \ z, z + a \in \mathcal{O}$$

Extend the map R to the whole plane using the first equation and denote the extension by R_1. Extend the map R using the second equation, and denote this extension by R_2. Being holomorphic extensions to $\mathbb{C}$ of the same function, R_1, R_2 are equal.

In particular the zero set of R is invariant under the group of transformations generated by $g_1(z) = \lambda z$, $g_2(z) = z + a$, which contains elements $g_n = z \pm \epsilon_n$, for some $\epsilon_n \to 0$. So as $R_{|_{h(J)}} = 0$, then $R_{|_{h(J) \pm \epsilon_n}} = 0$.

But this is impossible. Just take a point y to be one of the endpoints of a complementary interval of $h(J)$. Then $R(y - \epsilon_n)$ is positive starting with certain n. We come to contradiction.

So the harmonic measure is singular with respect to the packing measure $\prod_\delta$. As we know from Section 4, this implies

$$Hdim(\omega) < \delta = Hdim(J)$$

8 Appendix

The next lemma is a particular case of Proposition 8.3 of [ADU], where all necessary references are made to restore its proof. But we decided to include the proof here to make the paper as self contained as possible for the convenience of the reader.

Lemma 8.1 Let f be a rational function with parabolic point $p = 0$ whose Julia set J_f is contained in $\mathbb{R}$ and whose only Fatou component is the parabolic basin.

Let f_1^{-1} denote the inverse branch of f that sends the parabolic point to itself. Then for any closed interval $I \subset J_f$ close to 0

$$|f_1^{-n} I| \asymp \frac{1}{n^2} \text{ and } \forall x \in f_1^{-n} I \ \ |x| \asymp \frac{1}{n} \tag{8.1}$$

Proof: It is easier to carry out the estimates if we change the coordinates such that the parabolic point becomes ∞. If we denote the new function g without loss of generality we can assume that $(0, \infty) \subset F_g$, $0 \in J_g$ and that the expansion of g near infinity is

$$g(z) = z + 1 + \frac{A}{z} + \frac{B}{z^2} + \dots$$

or simply $g(z) = z + 1 + \frac{A}{z} + \theta_1(z) = z + 1 + \frac{A}{z} + \frac{B}{z^2} + \theta_1(z)$ where $|\theta_1(z)| \leq M_1 \frac{1}{z^2}$ and $|\theta_2(z)| \leq M_2 \frac{1}{z^3}$ for large values of z.

Let g_1^{-1} denote the inverse branch of g that sends infinity to itself. In this setting the statement we have to prove is: if $I \subset (-\infty, -R]$ for some large R, then

$$|g_1^{-n} I| \asymp 1 \quad \text{and} \quad \forall x \in g_1^{-n} I \ \ |x| \asymp n \tag{8.2}$$

We will start by proving the second estimate. For $z \in (-\infty, -R]$ let $g_1^{-1}(z) = w$. We get

$$w + 1 + \frac{A}{w} + \theta_1(w) = z \Rightarrow \; w^2 + w(z-1) + A = w\theta_1(w) \in [-1, 1]$$

Solving the double inequality we get that

$$w \leq \frac{z - 1 - \sqrt{(z-1)^2 - 4(A-1)}}{2} \text{ and } w \geq \frac{z - 1 - \sqrt{(z-1)^2 - 4(A+1)}}{2}$$

We will show that $\frac{z-1-\sqrt{(z-1)^2-4(A-1)}}{2} \leq z - \frac{1}{2}$ and $\frac{z-1-\sqrt{(z-1)^2-4(A+1)}}{2} \geq x - \frac{3}{2}$. These inequalities are equivalent to :

$$4(A-1) - 1 \leq -2z \quad \text{and } -2z \geq 1 - 4(A+1)$$

which are true for $z \in (-\infty, -R]$. Applying n times $z - \frac{3}{2} \leq g_1^{-1}(z) \leq z - \frac{1}{2}$ we get

$$x - \frac{3n}{2} \leq g_1^{-n}(z) \leq x - \frac{n}{2} \tag{8.3}$$

To get the first estimate of (8.2), let $I = [-x_0 - a_0, -x_0]$ and $g_1^{-n} I = [-x_n - a_n, -x_n]$. We have that:

$$g(-x_{n+1} - a_{n+1}) = -x_n - a_n = g(-x_{n+1}) - a_n$$

$$x_{n+1} + a_{n+1} - 1 + \frac{A}{x_{n+1} + a_{n+1}} - \frac{B}{(x_{n+1} + a_{n+1})^2} - \theta_3(x_{n+1} + a_{n+1}) =$$

$$= a_n + x_{n+1} - 1 + \frac{A}{x_{n+1}} - \frac{B}{x_{n+1}^2} - \theta_3(x_{n+1})$$

$$\Rightarrow \quad a_{n+1}[1 - \frac{A}{x_{n+1}(x_{n+1} + a_{n+1})} - \frac{B(2x_{n+1} + a_{n+1})}{x_{n+1}^2 (x_{n+1} + a_{n+1})^2}] \geq a_n - \frac{B'}{x_{n+1}^3}$$

Using the inequality (8.3) we get that

$$a_n + \frac{B''}{n^3} \geq a_{n+1}[1 + \frac{A(n)}{n^2}] \geq a_n - \frac{B''}{n^3} \tag{8.4}$$

for some positive constant B'' and for some $A(n)$ bounded away from zero and infinity. Let $c_n = \alpha \prod_{k=1}^{n-1} \frac{1}{1 + A(k)/k^2}$ where α small enough to guarantee $a_i \geq \frac{c_i}{\sqrt{i}}$ for $i = 1, \ldots n_0$. Notice that the product that defines c_n is bounded away from zero. We will prove by induction that $a_n \geq \frac{c_n}{\sqrt{n}}$.

Assume the inequality holds for n, then by (8.4)

$$a_{n+1} \geq \frac{\frac{c_n}{\sqrt{n}} - \frac{B''}{n^3}}{1 + \frac{A(n)}{n^2}}$$

It is enough to show that the last ratio is greater than $\frac{c_{n+1}}{\sqrt{n+1}}$. This is equivalent to :

$$\frac{1}{\sqrt{n}} - \frac{B''}{c_n n^3} \geq \frac{1}{\sqrt{n+1}} \ \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{n(n+1)(\sqrt{n}+\sqrt{n+1})} \geq \frac{B''}{c_n n^3}$$

which is true for $n \geq n_0$ if n_0 was selected large enough.

Use $a_n \geq \frac{c_n}{\sqrt{n}}$ in the inequality (8.4) to get

$$a_{n+1}[1 + \frac{A(n)}{n^2}] \geq a_n - \frac{c_0 a_n B''}{n^{2.5}}$$

therefore

$$a_{n+1} \geq a_n[1 - \frac{A'}{n^2} - \frac{c_0 B''}{n^{2.5}}].$$

This defines a convergent product, so $\inf a_n > 0$.

From the first inequality of (8.4) we immediately get $\sup a_n < \infty$.

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