Exercise 8.7

(a) Writing out the Kolmogorov's forward equations for ${}_tp_{30}^{00}$, we have

$$\frac{d}{dt}\,{}_tp_{30}^{00} = {}_tp_{30}^{01}\,\mu_{30+t}^{10} - \left[{}_tp_{30}^{00}\left(\mu_{30+t}^{01} + \mu_{30+t}^{02} + \mu_{30+t}^{02}\right)\right].$$

Notice that we need values for ${}_tp_{30}^{01}$ so that the corresponding KFE's satisfy

$$\frac{d}{dt}\,{}_tp_{30}^{01} = {}_tp_{30}^{00}\,\mu_{30+t}^{01} - \left[{}_tp_{30}^{01}\left(\mu_{30+t}^{10} + \mu_{30+t}^{12} + \mu_{30+t}^{13}\right)\right].$$

Using the Euler's method with step size $h = 1/12$, we have a system of equations to solve recursively:

$${}_{t+h}p_{30}^{00} = {}_tp_{30}^{00} + h\left\{{}_tp_{30}^{01}\,\mu_{30+t}^{10} - \left[{}_tp_{30}^{00}\left(\mu_{30+t}^{01} + \mu_{30+t}^{02} + \mu_{30+t}^{02}\right)\right]\right\}$$

and

$${}_{t+h}p_{30}^{01} = {}_tp_{30}^{01} + h\left\{{}_tp_{30}^{00}\,\mu_{30+t}^{01} - \left[{}_tp_{30}^{01}\left(\mu_{30+t}^{10} + \mu_{30+t}^{12} + \mu_{30+t}^{13}\right)\right]\right\}$$

with initial boundary conditions: ${}_0p_{30}^{00} = 1$ and ${}_0p_{30}^{01} = 0$.

The following values can easily be verified as solutions to these systems of equations (note that we also give the result for ${}_tp_{30}^{01}$ for convenience):

t	${}_tp_{30}^{00}$	${}_tp_{30}^{01}$	t	${}_tp_{30}^{00}$	${}_tp_{30}^{01}$	t	${}_tp_{30}^{00}$	${}_tp_{30}^{01}$
0	1.00000	0.00000	2	0.99514	0.00133	19	0.90729	0.02726
1/12	0.99981	0.00005	3	0.99248	0.00205	20	0.89721	0.03062
2/12	0.99962	0.00011	4	0.98964	0.00282	21	0.88621	0.03436
3/12	0.99942	0.00016	5	0.98661	0.00364	22	0.87418	0.03853
4/12	0.99923	0.00021	6	0.98337	0.00452	23	0.86106	0.04318
5/12	0.99903	0.00027	7	0.97988	0.00546	24	0.84673	0.04836
6/12	0.99884	0.00032	8	0.97614	0.00649	25	0.83109	0.05411
7/12	0.99864	0.00037	9	0.97210	0.00760	26	0.81404	0.06049
8/12	0.99844	0.00043	10	0.96774	0.00881	27	0.79548	0.06755
9/12	0.99824	0.00048	11	0.96303	0.01014	28	0.77530	0.07533
10/12	0.99804	0.00054	12	0.95793	0.01160	29	0.75340	0.08389
11/12	0.99784	0.00059	13	0.95240	0.01320	30	0.72969	0.09325
1	0.99764	0.00065	14	0.94639	0.01498	31	0.70409	0.10344
			15	0.93985	0.01694	32	0.67652	0.11447
			16	0.93273	0.01913	33	0.64696	0.12632
			17	0.92498	0.02155	34	0.61541	0.13895
			18	0.91652	0.02425	35	0.58188	0.15227

Thus, we see from the table that ${}_{35}p_{30}^{00} = 0.58188$. R code may be helpful to some:

```
x <- 30
muxt01 <- function(t){
a1 <- 4*10^(-4)
b1 <- 3.5*10^(-6)
c1 <- 0.14
out <- a1 + b1*exp(c1*(x+t))
out}
muxt02 <- function(t){
a2 <- 5*10^(-4)
b2 <- 7.6*10^(-5)
c2 <- 0.09
out <- a2 + b2*exp(c2*(x+t))
out}
muxt12 <- function(t){
muxt02(t)}
muxt32 <- function(t){
1.2*muxt02(t)}
muxt10 <- function(t){
0.1*muxt01(t)}
muxt03 <- function(t){
0.05*muxt01(t)}
muxt13 <- function(t){
muxt03(t)}
h <- 1/12
t <- seq(0,35,h)
mu01_seq <- muxt01(t)
mu02_seq <- muxt02(t)
mu12_seq <- muxt12(t)
mu32_seq <- muxt32(t)
mu10_seq <- muxt10(t)
mu03_seq <- muxt03(t)
mu13_seq <- muxt13(t)

tpx00 <- rep(0,length(t))
tpx01 <- rep(0,length(t))
# initial boundary conditions
tpx00[1] <- 1
tpx01[1] <- 0
hh <- 1
while (hh<length(t)) {
hh <- hh+1
tpx00[hh] <- tpx00[hh-1] + h*(tpx01[hh-1]*mu10_seq[hh-1] -
tpx00[hh-1]*(mu01_seq[hh-1]+mu02_seq[hh-1]+mu03_seq[hh-1]))
tpx01[hh] <- tpx01[hh-1] + h*(tpx00[hh-1]*mu01_seq[hh-1] -
tpx01[hh-1]*(mu10_seq[hh-1]+mu12_seq[hh-1]+mu13_seq[hh-1]))
}
```

(b) Let P denote the monthly premium so that the APV of future premiums at issue is given by

$$\text{APV}(\text{FP}_0) = P \times \ddot{a}^{(12)00}_{30:\overline{35}|} = P \times \sum_{k=0}^{35-(1/2)} v^{k/12} \, {}_{k/12}p^{00}_{30}.$$

The APV of future benefit payment immediately when becoming critically ill is

$$\text{APV}(\text{FBCI}_0) = 100000 \times \bar{A}^{03}_{30:\overline{35}|} = 100000 \times \int_0^{35} v^t \left({}_tp^{00}_{30} \, \mu^{03}_{30+t} + {}_tp^{01}_{30} \, \mu^{13}_{30+t} \right) dt.$$

For the death benefit part, the APV of future benefit immediately upon death is

$$\text{APV}(\text{FDB}_0) = 100000 \times \bar{A}^{02}_{30:\overline{35}|} = 100000 \times \int_0^{35} v^t \left({}_tp^{00}_{30} \, \mu^{02}_{30+t} + {}_tp^{01}_{30} \, \mu^{12}_{30+t} \right) dt.$$

Note that here no DB from critical illness because lump sum benefit would have been paid once reaching critical illness. Finally, for the sickness (or disability) benefit portion, the APV of the annuity income payable continuously while disabled is given by

$$\text{APV}(\text{FBDI}_0) = 75000 \times \bar{a}^{01}_{30:\overline{35}|} = 75000 \times \int_0^{35} v^t \, {}_tp^{01}_{30} dt.$$

(i) None of the APV formulas above can be explicitly calculated without resorting to some form of approximations: we have preferred to use repeated Simpson's rule (the R codes follow the results). Our numerical calculations show that

$$\begin{aligned} P &= \frac{\text{APV}(\text{FBCI}_0) + \text{APV}(\text{FDB}_0) + \text{APV}(\text{FBDI}_0)}{\ddot{a}^{(12)00}_{30:\overline{35}|}} \\ &= \frac{287.8026 + 8683.5 + 29660.94}{187.0252} = 206.5617. \end{aligned}$$

This agrees with the textbook answer, but not with the correction made in the Errata.

```
i <- 0.05
v <- 1/(1+i)
vt <- v^t

# APV of monthly premium of 1
ann.prem <- sum(vt*tpx00)

m <- 1
temp <- vt*(tpx00*mu03_seq+ tpx01*mu13_seq)
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
v <- v1
while (m < length(t)-2) {
m <- m+2
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
```

```
v <- v + v1
v}
# APV of critical illness benefit
APVFBCI <- 100000*v

m <- 1
temp <- vt*(tpx00*mu02_seq+ tpx01*mu12_seq)
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
v <- v1
while (m < length(t)-2) {
m <- m+2
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
v <- v + v1
v}
# APV of death benefit
APVFDB <- 100000*v

m <- 1
temp <- vt*tpx01
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
v <- v1
while (m < length(t)-2) {
m <- m+2
v1 <- (h/3)*(temp[m] + 4*temp[m+1] + temp[m+2])
v <- v + v1
v}
# APV of sickness benefit
APVFBDI <- 75000*v

# monthly premium
P <- (APVFBCI + APVFDB + APVFBDI)/ann.prem

> APVFBCI
[1] 287.8026
> APVFDB
[1] 8683.501
> APVFBDI
[1] 29660.94
> ann.prem
[1] 187.0252
> P
[1] 206.5617
```

(ii) Let P be the annual premium rate payable continuously in this case. Based on the

Thiele's differential equations, we have:

$$\frac{d}{dt}\,{}_tV^{(0)} = \delta\,{}_tV^{(0)} + P - \mu^{01}_{30+t}\big({}_tV^{(1)} - {}_tV^{(0)}\big) - \mu^{02}_{30+t}\big(100000 - {}_tV^{(0)}\big) - \mu^{03}_{30+t}\big(100000 - {}_tV^{(0)}\big)$$

and

$$\frac{d}{dt}\,{}_tV^{(1)} = \delta\,{}_tV^{(1)} - 75000 - \mu^{10}_{30+t}\big({}_tV^{(0)} - {}_tV^{(1)}\big) - \mu^{12}_{30+t}\big(100000 - {}_tV^{(0)}\big) - \mu^{13}_{30+t}\big(100000 - {}_tV^{(0)}\big).$$

A few comments are worth noting from these differential equations. First, note that there is no reserve needed when critically ill because the lump sum benefit would have been fully paid. Second, there is no premium P when in state 1 (or sick) but there is a sickness benefit payment of 75000. Finally, because of the approximation involving two equations in two unknowns, we adjust the Euler's method with the approximations

$$\frac{d}{dt}\,{}_tV^{(0)} \approx \frac{{}_tV^{(0)} - {}_{t-h}V^{(0)}}{h}$$

and

$$\frac{d}{dt}\,{}_tV^{(1)} \approx \frac{{}_tV^{(1)} - {}_{t-h}V^{(1)}}{h}$$

together with the boundary conditions ${}_{35}V^{(0)} = {}_{35}V^{(1)} = 0$. The approximations that follow used a step size of $h = 1/12$. This gives us $P = 2498.069$.

```
delta <- log(1+i)
n <- length(t)
Vt0 <- rep(0,n)
Vt1 <- rep(0,n)
# express the initial reserve as a function of premium rate P
V00 <- function(P) {
m <- n
while (m>1) {
m <- m-1
Vt0[m] <- Vt0[m+1] - h*(delta*Vt0[m+1] + P - mu01_seq[m+1]*(Vt1[m+1]-Vt0[m+1])
 - mu02_seq[m+1]*(100000-Vt0[m+1]) - mu03_seq[m+1]*(100000-Vt0[m+1]))
Vt1[m] <- Vt1[m+1] - h*(delta*Vt1[m+1] -75000 - mu10_seq[m+1]*(Vt0[m+1]
-Vt1[m+1]) - mu12_seq[m+1]*(100000-Vt1[m+1]) - mu13_seq[m+1]*(100000-Vt1[m+1]))
}
Vt0[1]}
# the following solves for P such that initial reserve for healthy V00 = 0
P <- uniroot(V00,c(0,75000))$root

> P
[1] 2498.069
```

(iii) Given the value of $P = 2498.069$, we then solve the Thiele's differential equations in part (ii) above with the boundary conditions. This leads us to the policy value at time 10 for a healthy life:

$${}_{10}V^{(0)} = 16925.88$$

```
Vt0 <- rep(0,n)
Vt1 <- rep(0,n)
m <- n
while (t[m]>10) {
m <- m-1
Vt0[m] <- Vt0[m+1] - h*(delta*Vt0[m+1] + P - mu01_seq[m+1]*(Vt1[m+1]-Vt0[m+1])
 - mu02_seq[m+1]*(100000-Vt0[m+1]) - mu03_seq[m+1]*(100000-Vt0[m+1]))
Vt1[m] <- Vt1[m+1] - h*(delta*Vt1[m+1] -75000 - mu10_seq[m+1]*(Vt0[m+1]
-Vt1[m+1]) - mu12_seq[m+1]*(100000-Vt1[m+1]) - mu13_seq[m+1]*(100000-Vt1[m+1]))
}
Vt010 <- Vt0[which(t==10)]

> Vt010
[1] 16925.88
```