Exercise 8.10

(a) In the insurance-with-lapses model that includes death after withdrawal, the probability that an active life age x will die within t years can be expressed as

$$\begin{aligned} {}_tp_x^{01} &= 1 - {}_tp_x^{00} - {}_tp_x^{02} \\ &= 1 - \exp\left[-\int_0^t \left(\mu_{x+s}^{01} + \mu_{x+s}^{02}\right) ds\right] - \int_0^1 {}_sp_x^{00}\, \mu_{x+s}^{02} \cdot \mathrm{e}^{-\int_0^{t-s} \mu_{x+s+w}^{21} dw} ds \\ &= 1 - \exp\left[-\int_0^t \left(\mu_{x+s}^{01} + \mu_{x+s}^{02}\right) ds\right] - \int_0^1 {}_sp_x^{00}\, \mu_{x+s}^{02} \cdot \mathrm{e}^{-\int_s^t \mu_{x+z}^{01} dz} ds, \end{aligned}$$

where in the last term we applied a change of variable $z = s + w$ and noted that $\mu_{x+z}^{21} = \mu_{x+z}^{01}$. Simplifying and combining terms, we get

$$\begin{aligned} {}_tp_x^{01} &= 1 - \mathrm{e}^{-\int_0^t \mu_{x+s}^{01} ds} \cdot \mathrm{e}^{-\int_0^t \mu_{x+s}^{02} ds} - \int_0^1 \mathrm{e}^{-\int_0^s \mu_{x+z}^{01} dz} \cdot \mathrm{e}^{-\int_0^s \mu_{x+z}^{02} dz}\, \mu_{x+s}^{02} \cdot \mathrm{e}^{-\int_s^t \mu_{x+z}^{01} dz} ds \\ &= 1 - \mathrm{e}^{-\int_0^t \mu_{x+s}^{01} ds} \cdot \mathrm{e}^{-\int_0^t \mu_{x+s}^{02} ds} - \mathrm{e}^{-\int_0^t \mu_{x+s}^{01} ds} \int_0^1 \mathrm{e}^{-\int_0^s \mu_{x+z}^{02} dz}\, \mu_{x+s}^{02} ds \\ &= 1 - \mathrm{e}^{-\int_0^t \mu_{x+s}^{01} ds} \left[\mathrm{e}^{-\int_0^t \mu_{x+s}^{02} ds} - \int_0^1 \mathrm{e}^{-\int_0^s \mu_{x+z}^{02} dz}\, \mu_{x+s}^{02} ds\right]. \end{aligned}$$

Consider the term inside the bracket. If we apply integration by parts, it is straightforward to see that

$$\int_0^1 \mathrm{e}^{-\int_0^s \mu_{x+z}^{02} dz}\, \mu_{x+s}^{02} ds = 1 - \mathrm{e}^{-\int_0^t \mu_{x+s}^{02} ds},$$

so that the term inside the bracket is equal to 1. Thus, we have

$$ {}_tp_x^{01} = 1 - \exp\left(-\int_0^t \mu_{x+s}^{01} ds\right),$$

which is exactly the same as that in a “alive-dead” model of the same intensity.

(b) This intuitively makes sense because for an active person, he can remain active until death or transition to the ‘lapsed’ state and then die from that state. And because the forces of mortality from either the ‘active’ or the ‘lapsed’ state are equal, then this is just like a single decrement model.