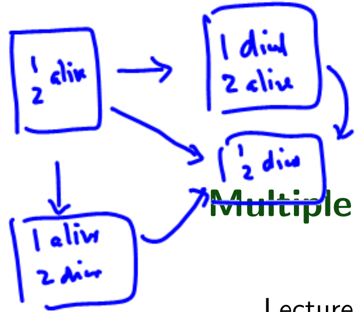
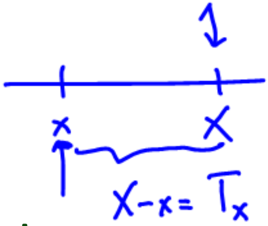
Multiple Life Models

Lecture: Weeks 9-10





$X, Y, Z,$

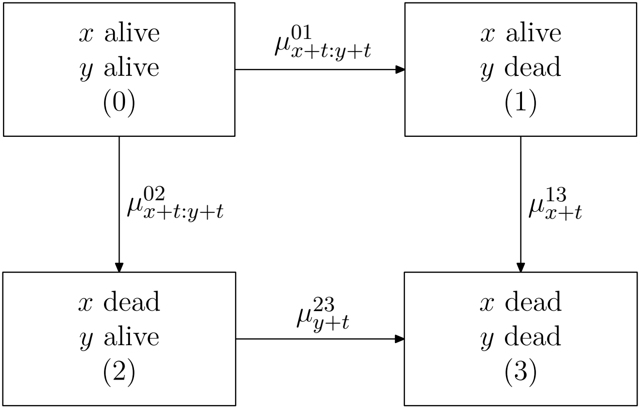
broken heart syndrome

(T_x, T_y)

Chapter summary

- Approaches to studying multiple life models:
  - define multiple states
  - traditional approach (use joint random variables)
- Statuses:
  - joint life status
  - last-survivor status
- Insurances and annuities involving multiple lives
  - evaluation using special mortality laws
- Simple reversionary annuities
- Contingent probability functions
- Dependent lifetime models
- Chapter 9 (Dickson et al.)

States in a joint life and last survivor model



Joint distribution of future lifetimes

Consider the case of two lives currently ages x and y with respective future lifetimes T_x and T_y.

- Joint cumulative dist. function: $F_{T_xT_y}(s,t) = \mathsf{Pr}[T_x \leq s, T_y \leq t]$
  - independence: $F_{T_xT_y}(s,t) = \mathsf{Pr}[T_x \leq s] \times \mathsf{Pr}[T_y \leq t] = F_x(s) \times F_y(t)$
- Joint density function: $f_{T_xT_y}(s,t) = \frac{\partial^2 F_{T_xT_y}(s,t)}{\partial s \partial t}$
  - independence: $f_{T_xT_y}(s,t) = f_x(s) \times f_y(t)$
- Joint survival dist. function: $S_{T_xT_y}(s,t) = \mathsf{Pr}[T_x > s, T_y > t]$
  - independence: $S_{T_xT_y}(s,t) = \mathsf{Pr}[T_x > s] \times \mathsf{Pr}[T_y > t] = S_x(s) \times S_y(t)$

Illustrative example 1

$\neq f_x(s) \cdot f_y(t)$

Consider the joint density expressed by

$$f_{T_xT_y}(s,t) = \frac{1}{64}(s+t), \quad \text{for } 0 < s < 4,\ 0 < t < 4.$$

1. Prove that T_x and T_y are not independent.
2. Calculate the covariance of T_x and T_y.
3. Evaluate the probability (x) outlives (y) by at least one year.

Solution to be discussed in lecture.

$T_x \geq T_y + 1$

(a) $f_X(s) = \int_0^4 \frac{1}{64}(s+t)\,dt = \frac{1}{16}(s+2), \quad 0 < s < 4$

$f_Y(t) = \int_0^4 \frac{1}{64}(s+t)\,ds = \frac{1}{16}(t+2), \quad 0 < t < 4$

$\frac{1}{16}(s+2)\,\frac{1}{16}(t+2) \neq \frac{1}{64}(s+t)$ not indp!

(b) $\mathrm{Cov}(T_x, T_y) = \overset{16/3}{E(T_x T_y)} - \overset{7/3}{E(T_x)}\,\overset{7/3}{E(T_y)} = -\frac{1}{9}$

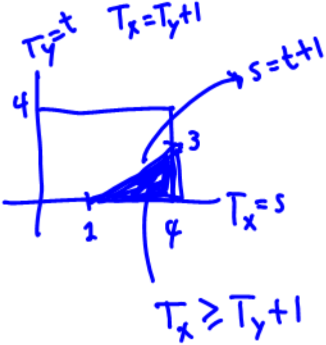
$E(T_x T_y) = \int_0^4 \int_0^4 s\,t\,\frac{1}{64}(s+t)\,ds\,dt = \frac{16}{3}$

$E(T_x) = E(T_y) = \int_0^4 s\,\frac{1}{16}(s+2)\,ds = \frac{7}{3}$

(c) $\Pr(T_x \geq T_y + 1) = \int_0^3 \int_{t+1}^4 \frac{1}{64}(s+t)\,ds\,dt$

$= .28125$

this is probability (x) outlive (y) by at least 1



(x)
(y) ages

$(T_x, T_y) \sim \begin{cases} f \\ F \\ S \end{cases}$

status

joint life $\min(T_x, T_y)$

lastsurvivor $+) \max(T_x, T_y)$

$T_x + T_y$

April 27-29 out of town

no classes

April 30 5-8 pm

Thursday Room C 304

The joint life status

This is a status that survives so long as all members are alive, and therefore fails upon the first death.

- Notation: (xy) for two lives (x) and (y)
- For two lives: $T_{xy} = \min(T_x, T_y)$
- Cumulative distribution function:

$$\begin{aligned} F_{T_{xy}}(t) &= {}_tq_{xy} = \Pr[\min(T_x, T_y) \le t] \\ &= 1 - \Pr[\min(T_x, T_y) > t] \\ &= 1 - \Pr[T_x > t, T_y > t] \\ &= 1 - S_{T_xT_y}(t, t) \\ &= 1 - {}_tp_{xy} \end{aligned}$$

where ${}_tp_{xy} = \Pr[T_x > t, T_y > t] = S_{T_{xy}}(t)$ is the probability that both lives (x) and (y) survive after t years.

The case of independence

- Alternative expression for the distribution function:

$$F_{T_{xy}}(t) = F_x(t) + F_y(t) - F_{T_xT_y}(t,t)$$

Pr(min$(T_x, T_y) \le t$) + Pr(max$(T_x, T_y) \le t$)

- In the case where T_x and T_y are independent:

$$\begin{aligned} {}_tp_{xy} &= \Pr[T_x > t, T_y > t] \\ &= \Pr[T_x > t] \times \Pr[T_y > t] \\ &= {}_tp_x \times {}_tp_y \end{aligned}$$

and

min + max = $T_x + T_y$

$${}_tq_{xy} = {}_tq_x + {}_tq_y - \underbrace{{}_tq_x \times {}_tq_y}_{{}_tq_{\overline{xy}}}$$

- Remember this (even in the case of independence):

$${}_tq_{xy} \neq {}_tq_x \times {}_tq_y$$

$$ {}_tq_{\overline{xy}} = {}_tq_x * {}_tq_y $$ independent,

joint life $\quad {}_tp_{xy} = {}_tp_x \cdot {}_tp_y,$

last survivor $\quad {}_tq_{\overline{xy}} = {}_tq_x \cdot {}_tq_y,$

independent

$$ {}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_x \cdot {}_tq_y, $$

The last-survivor status

This is a status that survives so long as there is at least one member alive, and therefore fails upon the last death.

- Notation: $(\overline{xy})$
- For two lives: $T_{\overline{xy}} = \max(T_x, T_y)$
- General relationship among T_{xy}, $T_{\overline{xy}}$, T_x, and T_y:

$$\begin{aligned} T_{xy} + T_{\overline{xy}} &= T_x + T_y \\ T_{xy} \cdot T_{\overline{xy}} &= T_x \cdot T_y \\ a^{T_{xy}} + a^{T_{\overline{xy}}} &= a^{T_x} + a^{T_y} \end{aligned}$$

  for any constant $a > 0$.

- For each outcome, note that T_{xy} is equal either T_x or T_y, and therefore, $T_{\overline{xy}}$ equals the other.

$${}_tp_{\overline{xy}} = Pr(\max(T_x, T_y) > t)$$

$$= Pr(T_x > t, T_y > t) = S_{T_x,T_y}(t,t)$$

prob that $(\overline{xy})$ survives to time t

~~both x and~~ either (x) or (y) reaches time t

$$1 - {}_tp_{\overline{xy}} = {}_tq_{\overline{xy}}$$

Distribution of $T_{\overline{xy}}$

$\cup$ = or

$\cap$ = and

Recall method of inclusion-exclusion of probability:
$\Pr[A \cup B] + \Pr[A \cap B] = \Pr[A] + \Pr[B]$.

- Choose events $A = \{T_x \leq t\}$ and $B = \{T_y \leq t\}$ so that

$$A \cup B = \{T_{xy} \leq t\} \text{ and } A \cap B = \{T_{\overline{xy}} \leq t\}.$$

- This leads us to the following useful relationships:

$$\begin{aligned} F_{T_{xy}}(t) + F_{T_{\overline{xy}}}(t) &= F_x(t) + F_y(t) \\ S_{T_{xy}}(t) + S_{T_{\overline{xy}}}(t) &= S_x(t) + S_y(t) \\ {}_tp_{xy} + {}_tp_{\overline{xy}} &= {}_tp_x + {}_tp_y \\ f_{T_{xy}}(t) + f_{T_{\overline{xy}}}(t) &= f_x(t) + f_y(t) \end{aligned}$$

- These relationships lead us to finding distributions of $T_{\overline{xy}}$, e.g.

$$F_{T_{\overline{xy}}}(t) = F_x(t) + F_y(t) - F_{T_{xy}}(t) = F_{T_xT_y}(t,t)$$

which is obvious from $F_{T_{\overline{xy}}}(t) = \Pr[T_x \leq t \cap T_y \leq t]$.

$$F_{T_{\overline{xy}}}(t) = \Pr\left(\max(T_x, T_y) \le t\right)$$

$$= \underbrace{\Pr\left(T_x \le t,\ T_y \le t\right)}_{F_{T_x, T_y}(t,t)}$$

$${}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_{\overline{xy}}$$

independence

$$= {}_tq_x\, {}_tp_y + {}_tq_y\, {}_tp_x + {}_tq_x\, {}_tq_y$$

Interpretation of probabilities

- Note that:
  - ${}_tp_{xy}$ is the probability that both lives (x) and (y) will be alive after t years.
  - ${}_tp_{\overline{xy}}$ is the probability that at least one of lives (x) and (y) will be alive after t years.
- In contrast:
  - ${}_tq_{xy}$ is the probability that at least one of lives (x) and (y) will be dead within t years.
  - ${}_tq_{\overline{xy}}$ is the probability that both lives (x) and (y) will be dead within t years.

Illustrative example 2

$_tq_x = t \cdot q_x$

$_tq_y = t \cdot q_y$

For independent lives (x) and (y), you are given:

$$q_x = 0.05 \text{ and } q_y = 0.10,$$

and

$$q_{x+1} = 0.06 \text{ and } q_{y+1} = 0.12.$$

Deaths are assumed to be uniformly distributed over each year of age. Calculate and interpret the following probabilities:

1. $_{0.75}q_{xy}$ $\rightarrow 1 - {}_{.75}p_{xy} = 1 - {}_{.75}p_x \cdot {}_{.75}p_y$ $= 0.1096875$
   $= \underbrace{1 - (1 - .75(.05))(1 - .75(.1))}$
2. $_{1.5}q_{\overline{xy}}$

Solution to be discussed in lecture.

$_{1.5}q_x \cdot {}_{1.5}q_y = (1 - p_x \cdot {}_{.5}p_{x+1})(1 - p_y \cdot {}_{.5}p_{y+1})$

$$= \underbrace{\left(1 - .95\left(1 - .5(.06)\right)\right)\left(1 - .90\left(1 - .5(.12)\right)\right)}$$

$$= \quad 0.987911$$

$$\mu_{x+t} = -\frac{d}{dt} \log {}_t p_x = \frac{-1}{{}_t p_x} \cdot \frac{d}{dt} {}_t p_x$$

$$= \frac{f_{T_x}(t)}{S_{T_x}(t)}$$

$$\mu_{x+t:y+t} = \frac{f_{T_{xy}}(t)}{S_{T_{xy}}(t)}$$

Force of mortality of T_{xy}

Define the force of mortality (similar manner to any random variable):

$$\mu_{x+t:y+t} = \frac{f_{T_{xy}}(t)}{1 - F_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{S_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{{}_tp_{xy}}.$$

- We can then write the density of T_{xy} as

$E[g(T_{xy})]$

$$f_{T_{xy}}(t) = {}_tp_{xy} \cdot \mu_{x+t:y+t}$$

$f_{T_x}(t) = {}_tp_x \, \mu_{x+t}$

- In the case of independence, we have:

$$\mu_{x+t:y+t} = \frac{{}_tp_x \cdot {}_tp_y(\mu_{x+t} + \mu_{y+t})}{{}_tp_x \cdot {}_tp_y} = \mu_{x+t} + \mu_{y+t}.$$

- The force of mortality of the joint life status is the sum of the individuals' force of mortality, when lives are independent.

Force of mortality for $T_{\overline{xy}}$

- The force of mortality for $T_{\overline{xy}}$ is defined as

$$\begin{aligned}\mu_{\overline{x+t:y+t}} &= \frac{f_{T_{\overline{xy}}}(t)}{1-F_{T_{\overline{xy}}}(t)} = \frac{f_{T_{\overline{xy}}}(t)}{S_{T_{\overline{xy}}}(t)} \\ &= \frac{f_x(t)+f_y(t)-f_{T_{xy}}(t)}{{}_tp_{\overline{xy}}} \\ &= \frac{{}_tp_x\cdot\mu_{x+t}+{}_tp_y\cdot\mu_{y+t}-{}_tp_{xy}\cdot\mu_{x+t:y+t}}{{}_tp_{\overline{xy}}}\end{aligned}$$

- Indeed we have the density of $T_{\overline{xy}}$ expressed as

${}_tp_{xy}\,\mu_{x+t:y+t}$

$T_{xy}+T_{\overline{xy}}=T_x+T_y$

$$f_{T_{\overline{xy}}}(t) = {}_tp_{\overline{xy}}\cdot\mu_{\overline{x+t:y+t}}.$$

- Check what this formula gives in the case of independence.

Insurance benefits - discrete

- Consider an insurance under which the benefit of \$1 is paid at the EOY of ending (failure) of status u.
- Status u could be any joint life or last survivor status e.g. xy, $\overline{xy}$. Then
  - the time at which the benefit is paid: $K_u + 1$
  - the present value (at issue) of the benefit: $Z = v^{K_u+1}$
  - APV of benefits: $\mathsf{E}[Z] = A_u = \sum_{k=0}^{\infty} v^{k+1} \cdot \mathsf{Pr}[K_u = k]$
  - variance: $\mathsf{Var}[Z] = {}^2A_u - (A_u)^2$

Insurance benefits - continuous

- Consider an insurance under which the benefit of $1 is paid immediately of ending (failure) of status u.
- Status u could be any joint life or last survivor status e.g. xy, $\overline{xy}$. Then
  - the time at which the benefit is paid: T_u
  - the present value (at issue) of the benefit: $Z = v^{T_u}$
  - APV of benefits: $\mathsf{E}[Z] = \bar{A}_u = \int_0^\infty v^t \cdot {}_tp_u \cdot \mu_{u+t} dt$
  - variance: $\mathsf{Var}[Z] = {}^2\bar{A}_u - (\bar{A}_u)^2$

Some illustrations

- For a joint life status (xy), consider whole life insurance providing benefits at the first death:

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k|}q_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k}p_{xy} \cdot q_{x+k:y+k}$$

$$\bar{A}_{xy} = \int_0^{\infty} v^t \cdot {}_{t}p_{xy} \cdot \mu_{x+t:y+t} dt$$

- For a last-survivor status $(\overline{xy})$, consider whole life insurance providing benefits upon the last death:

$A_{\overline{xy}} = A_x + A_y - A_{xy}$

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k|}q_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot \left({}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}\right)$$

$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

$$\bar{A}_{\overline{xy}} = \int_0^{\infty} v^t \cdot {}_{t}p_{\overline{xy}} \cdot \mu_{\overline{x+t:y+t}} dt$$

$$= \int_0^{\infty} v^t \left({}_{t}p_x \cdot \mu_{x+t} + {}_{t}p_y \cdot \mu_{y+t} - {}_{t}p_{xy} \cdot \mu_{x+t:y+t}\right) dt$$

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot \Pr\left(K_{\overline{xy}} = k+1\right)$$

$$\downarrow$$

$$_{k|}q_{\overline{xy}}$$

$$\neq$$

$$_{k|}q_x = {}_kp_x \, q_{x+k}$$

$$_{k|}q_{xy} = {}_kp_{xy} \, q_{x+k:y+k}$$

$$\cancel{_{k|}q_{\overline{xy}} \neq {}_kp_{\overline{xy}} \, q_{\overline{x+k:y+k}}}$$

$$A_{\overline{xy}} = A_x + A_y - A_{xy}$$

$$A_{xy} = A_x + A_y - A_{\overline{xy}}$$

- continued

- Useful relationships:

$$A_{xy} + A_{\overline{xy}} = A_x + A_y$$

$$\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y$$

Annuity benefits - discrete

Consider an n-year temporary life annuity-due on status u.

- Then
  - the present value (at issue) of the benefit: $Y = \begin{cases} \ddot{a}_{\overline{K_u+1}|}, & K_u < n \\ \ddot{a}_{\overline{n}|}, & K_u \geq n \end{cases}$
  - APV of benefits: $\mathsf{E}[Y] = \ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_u + \ddot{a}_{\overline{n}|} \cdot {}_n p_u$
  - variance: $\mathsf{Var}[Y] = \frac{1}{d^2}\left[{}^2A_{u:\overline{n}|} - \left(A_{u:\overline{n}|}\right)^2\right]$
- Other ways to write APV:

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_k p_u = \frac{1}{d}\left(1 - A_{u:\overline{n}|}\right).$$

Annuity benefits - continuous

Consider an annuity for which the benefit of \$1 is paid each year continuously for ∞ years so long as a status u continues.

- Then
  - the present value (at issue) of the benefit: $Y = \bar{a}_{\overline{T_u}|}$
  - APV of benefits: $\mathsf{E}[Y] = \bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|} \cdot {}_tp_u \cdot \mu_{u+t} dt = \int_0^\infty v^t\, {}_tp_u dt$
  - variance: $\mathsf{Var}[Y] = \frac{1}{\delta^2}\left[{}^2\bar{A}_u - \left(\bar{A}_u\right)^2\right]$
- Note that the identity $\delta \bar{a}_{\overline{T_u}|} + v^{T_u} = 1$ provides the connection between insurances and annuities.

Some illustrations

- For joint life status (xy), consider a whole life annuity providing benefits until the first death:

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{xy} \quad \text{and} \quad \bar{a}_{xy} = \int_0^{\infty} v^t \cdot {}_t p_{xy} dt$$

- For last survivor status $(\overline{xy})$, consider a whole life insurance providing benefits upon the last death:

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k \cdot {}_k p_{\overline{xy}} \quad \text{and} \quad \bar{a}_{\overline{xy}} = \int_0^{\infty} v^t \cdot {}_t p_{\overline{xy}} dt$$

- Useful relationships:

$$\begin{aligned} \ddot{a}_{xy} + \ddot{a}_{\overline{xy}} &= \ddot{a}_x + \ddot{a}_y \\ \bar{a}_{xy} + \bar{a}_{\overline{xy}} &= \bar{a}_x + \bar{a}_y \end{aligned}$$

Comparing benefits - annuities

Type of life annuity	Single life x	Joint life status xy	Last survivor status $\overline{xy}$
Whole life a-due	$\ddot{a}_x$	$\ddot{a}_{xy}$	$\ddot{a}_{\overline{xy}}$
Whole life a-immediate	a_x	a_{xy}	$a_{\overline{xy}}$
Temporary life a-due	$\ddot{a}_{x:\overline{n}\rceil}$	$\ddot{a}_{xy:\overline{n}\rceil}$	$\ddot{a}_{\overline{xy}:\overline{n}\rceil}$
Temporary life a-immediate	$a_{x:\overline{n}\rceil}$	$a_{xy:\overline{n}\rceil}$	$a_{\overline{xy}:\overline{n}\rceil}$
Whole life a-continuous	$\bar{a}_x$	$\bar{a}_{xy}$	$\bar{a}_{\overline{xy}}$
Temporary life a-continuous	$\bar{a}_{x:\overline{n}\rceil}$	$\bar{a}_{xy:\overline{n}\rceil}$	$\bar{a}_{\overline{xy}:\overline{n}\rceil}$

Comparing benefits - insurances

$\overline{xy}$

Type of life insurance	Single life x	Joint life status xy	Last survivor status $\overline{xy}$
Whole life - discrete	A_x	A_{xy}	$A_{\overline{xy}}$
Whole life - continuous	$\bar{A}_x$	$\bar{A}_{xy}$	$\bar{A}_{\overline{xy}}$
Term - discrete	$A^{\,1}_{x:\overline{n}\rvert}$	$A^{\,1}_{xy:\overline{n}\rvert}$	$A^{\,1}_{\overline{xy}:\overline{n}\rvert}$
Term - continuous	$\bar{A}^{\,1}_{x:\overline{n}\rvert}$	$\bar{A}^{\,1}_{xy:\overline{n}\rvert}$	$\bar{A}^{\,1}_{\overline{xy}:\overline{n}\rvert}$
Endowment - discrete	$A_{x:\overline{n}\rvert}$	$A_{xy:\overline{n}\rvert}$	$A_{\overline{xy}:\overline{n}\rvert}$
Endowment - continuous	$\bar{A}_{x:\overline{n}\rvert}$	$\bar{A}_{xy:\overline{n}\rvert}$	$\bar{A}_{\overline{xy}:\overline{n}\rvert}$
Pure endowment	$A_{x:\overset{1}{\overline{n}\rvert}}$ or ${}_nE_x$	$A_{xy:\overset{1}{\overline{n}\rvert}}$ or ${}_nE_{xy}$	$A_{\overline{xy}:\overset{1}{\overline{n}\rvert}}$ or ${}_nE_{\overline{xy}}$

(x) (y) $\quad (T_x, T_y) \sim f_{T_x,T_y}(t,s)$

<u>recall</u>: joint life status: (xy) $\quad T_{xy} = \min(T_x, T_y)$

last survivor status: $(\overline{xy})$ $\quad T_{\overline{xy}} = \max(T_x, T_y)$

$$\underbrace{P(T_{xy} > t)}_{{}_tp_{xy}} = P(T_x > t, T_y > t) = S(t,t)$$

$$P(T_{\overline{xy}} \leq t) = P(T_x \leq t, T_y \leq t) = F(t,t)$$

independence

$${}_tp_{xy} = {}_tp_x \cdot {}_tp_y \qquad {}_tq_{xy} = 1 - {}_tp_{xy} = 1 - {}_tp_x \, {}_tp_y$$
$$= {}_tq_x \, {}_tp_y + {}_tp_x \, {}_tq_y + {}_tq_x \, {}_tq_y$$

$$T_x + T_y = T_{xy} + T_{\overline{xy}}$$

$${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$$

$${}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}$$

$$f_{T_{xy}}(t) = {}_tp_{xy}\,\mu_{x+t:y+t}$$

indep

$$\stackrel{\checkmark}{=} {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$$

If T_x, T_y are independent,

$$f_{T_x,T_y}(t,s) = \underbrace{f_{T_x}(t)}_{{}_tp_x\,\mu_{x+t}} \cdot \underbrace{f_{T_y}(s)}_{{}_sp_y\,\mu_{y+s}}$$

$$F_{T_{xy}}(t) = P(T_{xy} \le t) = 1 - P(T_x > t, T_y > t)$$

$$\Downarrow \qquad = 1 - \underbrace{P(T_x > t)}_{{}_tp_x} \underbrace{P(T_y > t)}_{{}_tp_y}$$

$$\frac{\partial F_{T_{xy}}(t)}{\partial t} = f_{T_{xy}}(t) = -{}_tp_x \underbrace{\frac{\partial}{\partial t} {}_tp_y}_{-{}_tp_y \mu_{y+t}} - {}_tp_y \underbrace{\frac{\partial}{\partial t} {}_tp_x}_{-{}_tp_x \mu_{x+t}}$$

$$= {}_tp_x \, {}_tp_y \, (\mu_{x+t} + \mu_{y+t})$$

$$= {}_tp_{xy} \, (\mu_{x+t} + \mu_{y+t})$$

$$= {}_tp_{xy} \, \mu_{x+t:y+t}$$

u = status $\qquad xy$ or $\overline{xy}$

$$\bar{A}_u = E\left[v^{T_u}\right] \Longrightarrow \int_0^\infty v^t \cdot f_{T_u}(t)\, dt$$

$$A_u = E\left[v^{K_u+1}\right] \Rightarrow \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_u$$

$$\bar{A}_{xy} = E\left[v^{T_{xy}}\right] = \int_0^\infty v^t \cdot {}_tp_{xy}\, \mu_{x+t:y+t}\, dt$$

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Illustrative example 3

$T_{45} \sim \text{Uniform}\,(0, 60) \to \frac{1}{60}$

$T_{65} \sim \quad " \quad (0, 40) \sim \frac{1}{40}$

You are given:

- (45) and (65) have independent future lifetimes.
- Mortality for either life follows deMoivre's law with $\omega = 105$.
- $\delta = 5\%$

$f_{T_x T_y}(s,t) = \frac{1}{60} \cdot \frac{1}{40}, \quad \begin{array}{l} 0 \le s \le 60 \\ 0 \le t \le 40 \end{array}$

Calculate $\bar{A}_{\overline{45:65}}$.

$f_{T_{\overline{xy}}}(t)$

$$f(t) = \frac{1}{60}$$

$$F(t) = t/60$$

$$f_{T_{\overline{45:65}}}(t) = \frac{dF(t)}{dt},$$

$$F_{T_{\overline{45:65}}}(t) = \underbrace{Pr\left(\max\left(T_{45}, T_{65}\right) \leq t\right)}_{Pr(T_{45} \leq t)\, Pr(T_{65} \leq t)}$$

$$Pr(T_{45} \leq t) = \frac{t}{60}, \qquad Pr(T_{65} \leq t) = \frac{t}{40}$$

$$= \begin{cases} \frac{t}{60}\frac{t}{40}, & 0 \leq t \leq 40 \\ \frac{t}{60}, & 40 < t \leq 60 \\ 0, & \text{else} \end{cases}$$

$$f_{T_{\overline{45:65}}}(t) = \begin{cases} \frac{t}{1200}, & 0 \le t \le 40 \\ \frac{1}{60}, & 40 < t \le 60 \\ 0 \end{cases}$$

$$\underbrace{E\left[v^{T_{\overline{45:65}}}\right]} = \underbrace{\int_0^{40} v^t \frac{t}{1200}\,dt + \int_{40}^{60} v^t \frac{1}{60}\,dt}_{v^t = e^{-.05t}} + \phi \qquad \underbrace{\int t v^t\,dt}$$

$$\Downarrow$$

$$\bar{A}_{\overline{45:65}} = \underbrace{\frac{1}{1200}\underbrace{\int_0^{40} t v^t\,dt}_{237.5977} + \frac{1}{60}\underbrace{\int_{40}^{60} v^t\,dt}_{1.710964}}$$

$$= 0.2265141$$

$($

$\downarrow$ the A.P.V. of an insurance that pays \$1 upon the last death.

$\bar{A}_{45:65}$ = the A.P.V. of an insurance that pays \$1 upon the first death

$\Downarrow$

$$\bar{A}_{45} + \bar{A}_{65} - \bar{A}_{\overline{45:65}} \rightarrow 0.2265141$$

$$\int_0^{60} \frac{1}{60} v^t \, dt + \int_0^{40} \frac{1}{40} v^t \, dt$$

$>$ 0.2265741

It also works for annuities u = status -

e.g. whole life annuity-due on status (u)

$$PV = \ddot{a}_{\overline{K_u+1}|}$$

$$E[PV] = \ddot{a}_u = \underbrace{\sum_{K=0}^{\infty} \ddot{a}_{\overline{K+1}|} \cdot {}_{K|}q_u}$$

current payment technique

$$= \sum_{K=0}^{\infty} v^K {}_K p_u$$

e.g. joint life $\ddot{a}_{xy} = \sum_{K=0}^{\infty} v^K {}_K p_{xy}$

last survivor $\ddot{a}_{\overline{xy}} = \sum_{K=0}^{\infty} v^K {}_K p_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$

$\dot{a} \to A$

$$E\left[\ddot{a}_{\overline{K_n+1|}}\right] = E\left[\frac{1-v^{K_n+1}}{d}\right] = \frac{1-E\left[v^{K_n+1}\right]}{d}$$

e.g. $\ddot{a}_{xy} = \dfrac{1-A_{xy}}{d} \Rightarrow A_{xy} = 1 - d\,\ddot{a}_{xy}$

$\ddot{a}_{\overline{xy}} = \dfrac{1-A_{\overline{xy}}}{d} \Rightarrow A_{\overline{xy}} = 1 - d\,\ddot{a}_{\overline{xy}}$

$\Downarrow$

$A_{xy} = A_x + A_y - A_{\overline{xy}}$

$\ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{\overline{xy}}$

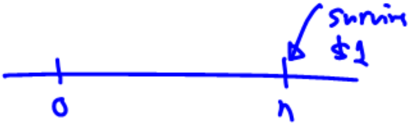
continuous also works.

$\ddot{a}_{xy} \neq \ddot{a}_x\,\ddot{a}_y$

$\ddot{a}_{xy}$ = the APV of an annuity that pays \$1 at the beginning of each year that both (x) and (y) are alive

$\ddot{a}_{\overline{xy}}$ = the APV of an annuity that pays \$1 at the beginning of each year that at least one of (x) or (y) is alive.

Pure endowment-



$${}_nE_x = v^n\,{}_np_x$$

$${}_nE_y = v^n\,{}_np_y$$

$${}_nE_{xy} = v^n\,{}_np_{xy} \neq {}_nE_x \cdot {}_nE_y$$

$$= (1+i)^n\,{}_nE_x \cdot {}_nE_y$$ <u>if independent!</u>

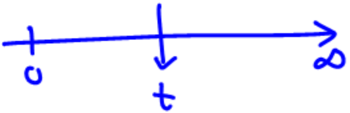
$${}_nE_{\overline{xy}} = {}_nE_x + {}_nE_y - {}_nE_{xy}$$

Contingent functions

- It is possible to compute probabilities, insurances and annuities based on the failure of the status that is contingent on the order of the deaths of the members in the group, e.g. (x) dies before (y).
- These are called contingent functions.
- Consider the probability that (x) fails before (y) - assuming independence:

$$
\begin{aligned}
\Pr[T_x < T_y] &= \int_0^\infty f_{T_x}(t) \cdot S_{T_y}(t)\,dt \\
&= \int_0^\infty {}_tp_x\,\mu_{x+t} \cdot {}_tp_y\,dt \\
&= \int_0^\infty {}_tp_{xy}\,\mu_{x+t}\,dt
\end{aligned}
$$

$_{\infty}q^{1}_{xy}$



${}_tp_x\,\mu_{x+t} \cdot {}_tp_y$

- The actuarial symbol for this is $_{\infty}q^{1}_{xy}$. It should be obvious this is the same as $_{\infty}q_{xy}^{\;\,2}$.

$$\underline{{}_{\infty}q_{x\overset{2}{y}} = {}_{\infty}q_{\overset{1}{x}y}} \longrightarrow \text{but } \underbrace{{}_{n}q_{x\overset{2}{y}} \neq {}_{n}q_{\overset{1}{x}y}}$$

$$\Downarrow$$

$${}_{\infty}q_{\overset{2}{x}y} = \underbrace{{}_{\infty}q_{x\overset{1}{y}}} \longrightarrow \underbrace{\int_0^{\infty} {}_tp_{xy}\,\mu_{y+t}\,dt}$$

?? independent

$$\int_0^{\infty} {}_tq_y\,{}_tp_x\,\mu_{x+t}\,dt$$

$$\underbrace{{}_{n}q_{\overset{1}{x}y}} + \underbrace{{}_{n}q_{\overset{2}{x}y}} = {}_{n}p_{xy} - {}_{n}q_{\overline{xy}}$$

?? independent

$$\underbrace{\int_0^n {}_tp_y\,{}_tp_x\,\mu_{x+t}\,dt + \int_0^n {}_tq_y\,{}_tp_x\,\mu_{x+t}\,dt}_{{}_nq_x}$$

Insurances on contingent lives -

Reminder

April 8

2 formula sheets

- continued

- The probability that (x) dies before (y) and within n years is given by

$$ {}_{n}q_{\overset{1}{x}y} = \int_0^n {}_{t}p_{xy}\,\mu_{x+t}\,dt. $$

- Similarly, we have the probability that (y) dies before (x) and within n years:

$$ {}_{n}q_{x\overset{1}{y}} = \int_0^n {}_{t}p_{xy}\,\mu_{y+t}\,dt. $$

- It is easy to show that ${}_{n}q_{\overset{1}{x}y} + {}_{n}q_{x\overset{1}{y}} = {}_{n}q_{xy}$.
- One can similarly define and interpret the following: ${}_{n}q_{\overset{2}{x}y}$ and ${}_{n}q_{x\overset{2}{y}}$, and show that

$$ {}_{n}q_{\overset{2}{x}y} + {}_{n}q_{x\overset{2}{y}} = {}_{n}q_{\overline{xy}}. $$

$${}_nq_{\overset{2}{x}y} = \cancel{\int_0^n {}_tp_{xy}\,\mu_{x+t}\,dt} \quad \int_0^n \overset{1-{}_tp_y}{{}_tq_y'}\;{}_tp_x\,\mu_{x+t}\,dt$$

$${}_nq_{x\overset{2}{y}} = \cancel{\int_0^n {}_tp_{xy}\,\mu_{y+t}\,dt} \quad \int_0^n \overset{1-{}_tp_x}{{}_tq_x'}\;{}_tp_y\,\mu_{y+t}\,dt$$

$$\underbrace{{}_nq_x + {}_nq_y - {}_nq_{xy}}_{{}_nq_{\overline{xy}}}$$

$${}_nq_{\overset{1}{x}y\overset{2}{z}}$$

$${}_nq_{xy^{1}z} =$$

$$\int_0^n {}_tp_{xyz}\,\mu_{y+t}\,dt$$

$${}_nq_{\overset{1}{x}y\overset{2}{z}}$$

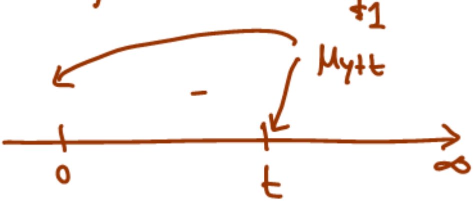
$$\int_0^n {}_tq_x\,{}_tp_y\,{}_tp_z\,\mu_{z+t}$$

$\underbrace{\bar{A}_{x\overset{2}{y}}}$

insurance pays at the moment (y) dies provided (y) is ~~predeceased~~ by (x)

predeceased

(y) dies after (x)

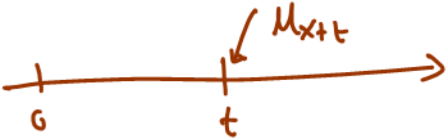


$(1-{}_tp_x)$

$$\int_0^\infty v^t\, {}_tq_x\, {}_tp_y\, \mu_{y+t}\, dt$$

$$\bar{A}_{x\overset{2}{y}} = \bar{A}_y - \bar{A}_{x\overset{1}{y}} \Rightarrow \bar{A}_{x\overset{1}{y}} + \bar{A}_{x\overset{2}{y}} = \bar{A}_y$$

$\bar{A}_{xy}^{\;\,2}$



$$\int_0^\infty v^t \bar{A}_{y+t}\, {}_tp_y\, {}_tp_x\, \mu_{x+t}\, dt$$

$$= \int_0^\infty v^t\, {}_tq_x\, {}_tp_y\, \mu_{y+t}\, dt$$

$$\int_0^\infty v^{s+y+t}\, {}_sp_{y+t}\, \mu_{y+t+s}\, ds$$

$$PV = \begin{cases} v^{T_y}, & T_y \geq T_x \\ 0, & T_y < T_x \end{cases}$$

$\bar{A}_{x\overset{2}{y}}$,

$$= v^{T_y} \cdot I(T_y \geq T_x)$$

$$E[PV] = E[v^{T_y} \underbrace{I(T_y \geq T_x)}] \longrightarrow \bar{A}_{x\overset{2}{y}}$$

$$+\ E[v^{T_y} I(T_y \leq T_x)] \rightarrow \bar{A}^{2}_{xy} \text{ or } \bar{A}_{x\overset{1}{y}}$$

$$E[v^{T_y} \cdot 1] = \bar{A}_y$$

Illustrative example 4

An insurance of \$1 is payable at the moment of death of (y) if predeceased by (x), i.e. if (y) dies after (x). The actuarial present value (APV) of this insurance is denoted by $\bar{A}_{x\overset{2}{y}}$. Assume (x) and (y) are independent.

1. Give an expression for the present value random variable for this insurance.
2. Show that
$$\bar{A}_{x\overset{2}{y}} = \bar{A}_y - \bar{A}_{x\overset{1}{y}}.$$
3. Prove that
$$\bar{A}_{x\overset{2}{y}} = \int_0^\infty v^t \bar{A}_{y+t}\, {}_tp_{xy}\, \mu_{x+t} dt,$$
and interpret this result.

Take the case of constant forces stop here

$T_x \sim$ exponential with $\mu_x = .02$

$T_y \sim$ " " $\mu_y = .01$

$$\bar{A}_x = \frac{\mu_x}{\mu_x + \delta}$$

Derive $\bar{A}_{xy}$ $\bar{a}_{xy}$

$\bar{A}_{\overline{xy}}$ $\bar{a}_{\overline{xy}}$

A_{xy} $\ddot{a}_{xy}$

$A_{\overline{xy}}$ $\ddot{a}_{\overline{xy}}$

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

$$\bar{A}_{xy} = 1 - \delta \bar{a}_{xy}$$

Single life / discrete given constant force.

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x \quad {}_{k}p_x \, q_{x+k} \to e^{-\mu_x k}(1-e^{-\mu_x})$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_{k}p_x$$

$v^k \to e^{-\delta k}$, ${}_{k}p_x \to e^{-\mu_x k}$

$$\underbrace{e^{-\delta k} \, e^{-\mu_x k}}$$

$$\sum_{k=0}^{\infty} e^{-(\mu_x+\delta)k} = \frac{1}{1-e^{-(\mu_x+\delta)}}$$

$\ddot{a}_{xy}$

$\ddot{a}_{\overline{xy}}$

A_{xy}

$A_{\overline{xy}}$

$$A_x = 1 - d\,\ddot{a}_x$$

Exam 2: STOP HERE

(x), (y)

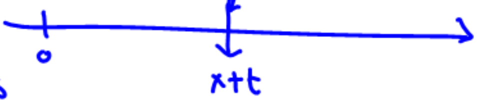
r.a. after death of y

continuous $1 per year.

Reversionary annuity

pay ~~once~~ commences upon death of the other.

$$v^t \; {}_tp_{xy} \; \mu_{y+t} \cdot \bar{a}_{x+t}$$



$$\bar{a}_{y|x} = \int_0^\infty v^t \, {}_tp_{xy} \, \mu_{y+t} \, \bar{a}_{x+t} \, dt$$

CPT'

$$\vdots$$

$$= \int_0^\infty v^t \, {}_tp_x (1 - {}_tp_y) \, dt = \bar{a}_x - \bar{a}_{xy}$$

Reversionary annuities

A reversionary annuity is an annuity which commences upon the failure of a given status (u) if a second status (v) is then alive, and continues thereafter so long as status (v) remains alive.

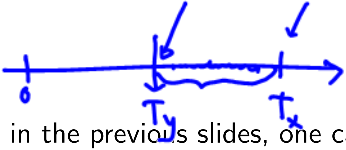
- Consider the simplest form: an annuity of \$1 per year payable continuously to a life now aged x, commencing at the moment of death of (y) - briefly annuity to (x) after (y).
- APV for this reversionary annuity:

$$\bar{a}_{y|x} = \int_0^\infty v^t \, {}_tp_{xy} \mu_{y+t} \bar{a}_{x+t} dt.$$

- One can show the more intuitive formula (using current payment technique):

$$\bar{a}_{y|x} = \int_0^\infty v^t \, {}_tp_x \left(1 - {}_tp_y\right) dt = \bar{a}_x - \bar{a}_{xy}.$$

Present value random variable



- For the reversionary annuity considered in the previous slides, one can also write the present-value random variable at issue as:

$E[Z] = \bar{a}_{y|x}$

$= \underline{\underline{\bar{a}_x - \bar{a}_{xy}}}$

$$
\begin{aligned}
Z &= \begin{cases} {}_{T_y|}\bar{a}_{\overline{T_x - T_y}|}, & T_y \leq T_x \\ 0, & T_y > T_x \end{cases} \\
&= \begin{cases} \bar{a}_{\overline{T_x}|} - \bar{a}_{\overline{T_y}|}, & T_y \leq T_x \\ 0, & T_y > T_x \end{cases} \\
&= \bar{a}_{\overline{T_x}|} - \bar{a}_{\overline{T_{xy}}|}.
\end{aligned}
$$

$T_y > T_x \rightarrow \bar{a}_{\overline{T_x}|} - \bar{a}_{\overline{T_x}|}$

Can you explain the last line?

- By taking the expectation of Z, we clearly have $\bar{a}_{y|x} = \bar{a}_x - \bar{a}_{xy}$.

Reversionary annuities - discrete

- In general, an annuity to any status (u) after status (v) is

$$a_{v|u} = a_u - a_{uv}$$

where a is any annuity which takes discrete, continuous, or payable m times a year.

- Consider the discrete form of reversionary annuity: \$1 per year payable to a life now aged x, commencing at the EOY of death of (y).
- APV for this reversionary annuity:

$$a_{y|x} = \sum_{k=1}^{\infty} v^k \, {}_k p_x \left(1 - {}_k p_y\right) = a_x - a_{xy}.$$

- If (v) is the term-certain $(\overline{n}|)$ and (u) is the single life (x), then

${}_{n|}a_x$ $$a_{\overline{n}|x} = a_x - a_{x:\overline{n}|}$$

which is indeed a single-life deferred annuity.

(x), (y), (z)

$$\bar{a}_{y|xz} = \bar{a}_{xz} - \bar{a}_{xyz}$$

you pay annuity starting upon death of (y) until the first death of (x), (z).

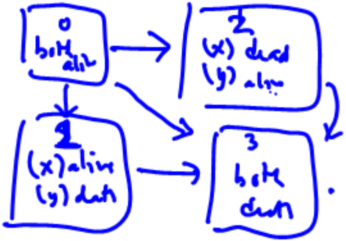
$$\bar{a}_{y|\overline{xz}} = \bar{a}_{\overline{xz}} - \bar{a}_{y:\overline{xz}}$$

should be able to interpret

Rewrite:

${}_tp_{xy}$

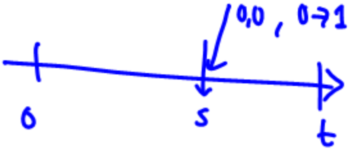
$${}_tp_{xy} = {}_tp^{00}_{xy}$$



$${}_tp_{\overline{xy}} = 1 - {}_tp^{03}_{xy} = {}_tp^{01}_{xy} + {}_tp^{02}_{xy} + {}_tp^{00}_{xy}$$

Back to multiple state framework

Translating the probabilities/forces earlier defined, the following should now be straightforward to verify:



- $_tp_{xy} = {_tp_{xy}^{00}}$
- $_tq_{xy} = {_tp_{xy}^{01}} + {_tp_{xy}^{02}} + {_tp_{xy}^{03}}$
- $_tp_{\overline{xy}} = {_tp_{xy}^{00}} + {_tp_{xy}^{01}} + {_tp_{xy}^{02}}$
- $_tq_{\overline{xy}} = {_tp_{xy}^{03}}$
- $_tq_{\overline{xy}} = {_tp_{xy}^{03}}$
- $_tq_{xy}^{1} = \int_0^t {_sp_{xy}^{00}}\, \mu_{x+s:y+s}^{02} ds$
- $_tq_{xy}^{2} = \int_0^t {_sp_{xy}^{01}}\, \mu_{x+s}^{13} ds$

Annuities

In terms of the annuity functions, the following should also be straightforward to verify:

- $\bar{a}_{xy} = \bar{a}_{xy}^{00} = \int_0^\infty \mathrm{e}^{-\delta t}\, {}_tp_{xy}^{00} dt$
- $\bar{a}_{\overline{xy}} = \bar{a}_{xy}^{00} + \bar{a}_{xy}^{01} + \bar{a}_{xy}^{02} = \int_0^\infty \mathrm{e}^{-\delta t}\left({}_tp_{xy}^{00} + {}_tp_{xy}^{01} + {}_tp_{xy}^{02}\right) dt$
- $\bar{a}_{x|y} = \bar{a}_{xy}^{02} = \int_0^\infty \mathrm{e}^{-\delta t}\, {}_tp_{xy}^{02} dt$

The following also holds true (easy to verify):

- $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$
- $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$

Insurances

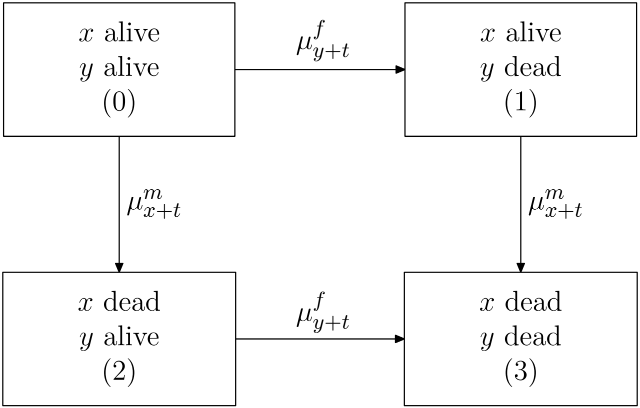
In terms of insurance functions, the following should also be straightforward to verify:

- $\bar{A}_{xy} = \int_0^\infty \mathrm{e}^{-\delta t}\, {}_tp_{xy}^{00} \big(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02}\big) dt$
- $\bar{A}_{\overline{xy}} = \int_0^\infty \mathrm{e}^{-\delta t} \big({}_tp_{xy}^{01}\, \mu_{x+t}^{13} + {}_tp_{xy}^{02}\, \mu_{y+t}^{23}\big) dt$
- $\bar{A}_{xy}^{1} = \int_0^\infty \mathrm{e}^{-\delta t}\, {}_tp_{xy}^{00}\, \mu_{x+t:y+t}^{02} dt$
- $\bar{A}_{xy}^{2} = \int_0^\infty \mathrm{e}^{-\delta t}\, {}_tp_{xy}^{01}\, \mu_{x+t}^{13} dt$

The following also holds true (easy to verify):

- $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$ and $\bar{a}_{xy} = \dfrac{1}{\delta}\big(1 - \bar{A}_{xy}\big)$
- $\bar{A}_{xy}^{1} + \bar{A}_{xy}^{2} = \bar{A}_x$

The case of independence



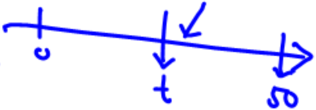
Illustrative example 5

$T_x \sim U(0,50)$ $T_y \sim U(0,50)$

Suppose that the future lifetimes, T_x and T_y, of a husband and wife, respectively are independent and each is uniformly distributed on $[0, 50]$. Assume $\delta = 5\%$.

1. A special insurance pays $1 upon the death of the husband, provided that he dies first. Calculate the actuarial present value for this insurance and the variance of the present value.
2. An insurance pays $1 at the moment of the husband's death if he dies first and $2 if he dies after his wife. Calculate the APV of the benefit for this insurance.
3. An insurance pays $1 at the moment of the husband's death if he dies first and $2 at the moment of the wife's death if she dies after her husband. Calculate the APV of the benefit for this insurance.

① $APV = \int_0^{50} v^t \underbrace{{}_tp'_x \, \mu_{x+t}} \, {}_tp'_y \, dt$



$= \int_0^{50} e^{-.05t} \frac{1}{50} \cdot (1 - t/50) \, dt$

$= .2531336$

$$\text{Variance} = APV@2\delta - (APV)^2 \qquad 2\delta$$

$$= \underbrace{\int_0^{50} e^{-.10t} \frac{1}{50}\left(1 - \frac{t}{50}\right) dt}_{.1602695} - (.2531336)^2$$

$= .0961928$

$(1 - {}_tp_y)$

② $APV = \underbrace{1 \cdot \int_0^{50} v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, dt} + 2\int_0^{50} v^t \, {}_tp_x \, \mu_{x+t} \, {}_tq_y \, dt$

$= 2\int_0^{50} v^t \, {}_tp_x \, \mu_{x+t} \, dt - APV(@1)$

$$\underbrace{\qquad\qquad}_{e^{-\delta t}\frac{1}{50}}$$

$T_x \sim U(0,50)$

$T_y \sim U(0,50)$

$= .0481984$

(3) $APV = APV(@\,1) + 2\underbrace{\int_0^{\infty} v^t \, \underbrace{{}_tp_y \mu_{y+t}}\, {}_tp_x \, dt}_{APV(@\,1)}$

$= 3(.2531336) = \underline{\underline{0.7594008}}$

Illustrative example 6

For a husband and wife with ages x and y, respectively, you are given:

- $\mu_{x+t} = 0.02$ for all $t > 0$
- $\mu_{y+t} = 0.01$ for all $t > 0$
- $\delta = 0.04$

assume independent

1. Calculate $\bar{a}_{xy:\overline{20|}}$ and $\bar{a}_{\overline{xy}:\overline{20|}}$.
2. Rewrite this problem in a multiple state framework and solve (1) within this framework.

$$\bar{a}_{xy:\overline{20}|} = \int_0^{20} v^t \, {}_tp_{xy} \, dt$$

$$v^t = e^{-.04t}$$

$$ {}_tp_{xy} = {}_tp_x \, {}_tp_y = e^{-\mu_x t} \, e^{-\mu_y t} = e^{-.02t} \, e^{-.01t}$$

$$= \boxed{10.7629}$$

$$\bar{a}_{\overline{xy}:\overline{20}|} = \bar{a}_{x:\overline{20}|} + \bar{a}_{y:\overline{20}|} - \bar{a}_{xy:\overline{20}|}$$

$$\bar{a}_{x:\overline{20}|} = \int_0^{20} e^{-.04t} \, {}_tp_x \, dt, \quad {}_tp_x = e^{-.02t}$$

$$\bar{a}_{y:\overline{20}|} = \int_0^{20} e^{-.04t} \, {}_tp_y \, dt, \quad {}_tp_y = e^{-.01t}$$

$$= 13.52627$$

Illustrative example 7: SOA Fall 2013 Question # 2

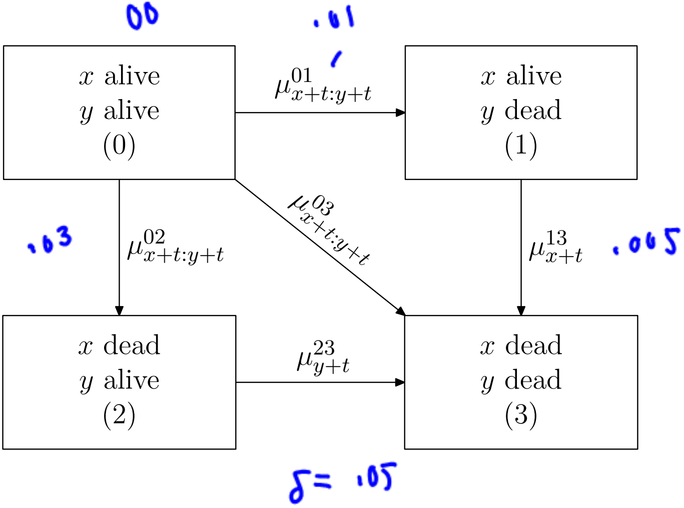
For (x) and (y) with independent future lifetimes, you are given:

- $\bar{a}_x = 10.06$
- $\bar{a}_y = 11.95$
- $\bar{a}_{\overline{xy}} = 12.59$ ✓
- $\bar{A}_{x\overset{1}{y}} = 0.09$
- $\delta = 0.07$

SOA
done in practice exam Q#9
similar one in test Q#9

Calculate $\bar{A}_{\overset{1}{x}y}$.

The model with a common shock

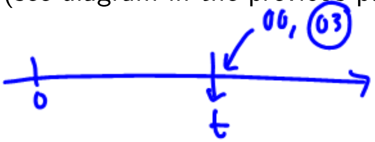


Illustrative example 8: SOA Spring 2014 Question # 7

The joint mortality of two lines (x) and (y) is being modeled as a multiple state model with a common shock (see diagram in the previous page).

You are given:

- $\mu^{01} = 0.010$
- $\mu^{02} = 0.030$ .045
- $\mu^{03} = 0.005$
- $\delta = 0.05$



A special joint whole life insurance pays 1000 at the moment of simultaneous death, if that occurs, and zero otherwise.

Calculate actuarial present value of this insurance.

$$1000 * \int_0^\infty \underset{.05}{e^{-\delta t}} \, \underbrace{{}_tp^{00}}_{e^{-.045t}} \, \underset{.005}{\mu^{03}} \, dt$$

$$= 1000\left(\frac{.005}{.095}\right)$$

$$\underline{\underline{52.631578}}$$