Multiple Decrement Models

Lecture: Weeks 8-9

Lecture summary

- Multiple decrement model - expressed in terms of multiple state model
- Multiple Decrement Tables (MDT)
  - several causes of decrement
  - probabilities of decrement
  - forces of decrement
- The Associated Single Decrement Tables (ASDT)
- Uniform distribution of decrements
  - in the multiple decrement context
  - in the associated single decrement context
- Chapter 8 (DHW), Sections 8.8-8.12

Examples of multiple decrement models

- Multiple decrement models are extensions of standard mortality models whereby there is simultaneous operation of several causes of decrement.
- A life fails because of one of these decrements.
- Examples include:
  - life insurance contract is terminated because of death/survival or withdrawal (lapse).
  - an insurance contract provides coverage for disability and death, which are considered distinct claims.
  - life insurance contract pays a different benefit for different causes of death (e.g. accidental death benefits are doubled).
  - pension plan provides benefit for death, disability, employment termination and retirement.

Introducing notation

age x	no. of lives $\ell_x^{(\tau)}$	heart disease $d_x^{(1)}$	accidents $d_x^{(2)}$	other causes $d_x^{(3)}$
50	$4,832,555$	$5,168$	$1,157$	$4,293$
51	$4,821,927$	$5,363$	$1,206$	$5,162$
52	$4,810,206$	$5,618$	$1,443$	$5,960$
53	$4,797,185$	$5,929$	$1,679$	$6,840$
54	$4,782,727$	$6,277$	$2,152$	$7,631$

- Conventional notation:
  - $\ell_x^{(\tau)}$ represents the surviving population present at exact age x.
  - $d_x^{(j)}$ represents the number of lives exiting from the population between ages x and $x+1$ due to decrement j.
  - It is also conventional to denote the total number of exits by all modes between ages x and $x+1$ by $d_x^{(\tau)}$ i.e.

$$d_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)}$$

where m is the total number of possible decrements, and therefore, $d_x^{(\tau)} = \ell_x^{(\tau)} - \ell_{x+1}^{(\tau)}$.

Probabilities of decrement

- The probability that a life (x) will leave the group within one year as a result of decrement j:

$$q_x^{(j)} = d_x^{(j)} / \ell_x^{(\tau)}.$$

- The probability that (x) will leave the group (regardless of decrement):

$$q_x^{(\tau)} = d_x^{(\tau)} / \ell_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)} / \ell_x^{(\tau)} = \sum_{j=1}^{m} q_x^{(j)}.$$

- The probability that (x) will remain in the group for at least one year:

$$p_x^{(\tau)} = 1 - q_x^{(\tau)} = \ell_{x+1}^{(\tau)} / \ell_x^{(\tau)} = (\ell_x^{(\tau)} - d_x^{(\tau)}) / \ell_x^{(\tau)}.$$

- continued

- We also have the probability of remaining in the group after n years

$$ {}_np_x^{(\tau)} = \ell_{x+n}^{(\tau)}/\ell_x^{(\tau)} = p_x^{(\tau)} \cdot p_{x+1}^{(\tau)} \cdots p_{x+n-1}^{(\tau)}. $$

  and the complement

$$ {}_nq_x^{(\tau)} = 1 - {}_np_x^{(\tau)}. $$

- The number of failures due to decrement j over the interval $(x, x+n]$ is

$$ {}_nd_x^{(j)} = \sum_{t=0}^{n-1} d_{x+t}^{(j)}. $$

- These relationships should be straightforward to follow:

$$ \begin{aligned} {}_nd_x^{(j)} &= \ell_x^{(\tau)} \cdot {}_nq_x^{(j)} \\ {}_nd_x^{(\tau)} &= \ell_x^{(\tau)} \cdot {}_nq_x^{(\tau)} \end{aligned} $$

Illustration of Multiple Decrement Table

Expand Multiple Decrement Table (MDT) into:

x	$\ell_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$
50	4,832,555	5,168	1,157	4,293	0.00107	0.00024	0.00089	0.00220	0.99780
51	4,821,927	5,363	1,206	5,162	0.00111	0.00025	0.00107	0.00243	0.99757
52	4,810,206	5,618	1,443	5,960	0.00117	0.00030	0.00124	0.00271	0.99729
53	4,797,185	5,929	1,679	6,840	0.00124	0.00035	0.00143	0.00301	0.99699
54	4,782,727	6,277	2,152	7,631	0.00131	0.00045	0.00160	0.00336	0.99664

Illustrative problems

Using the previously given multiple decrement table, compute and interpret the following:

1. ${}_2d_{51}^{(3)}$
2. ${}_3p_{50}^{(\tau)}$
3. ${}_2q_{53}^{(1)}$
4. ${}_{2|2}q_{50}^{(2)}$

Total force of decrement

- The total force of decrement at age x is defined as

$$\mu_x^{(\tau)} = \lim_{h \to 0} \frac{1}{h}\, {}_h q_x^{(\tau)} = -\frac{1}{\ell_x^{(\tau)}} \frac{d}{dx} \ell_x^{(\tau)} = -\frac{d}{dx} \log \ell_x^{(\tau)}$$

- Therefore, analogous to the single decrement table, we have

$${}_t p_x^{(\tau)} = \exp -\left(\int_0^t \mu_{x+s}^{(\tau)} ds \right)$$

and

$$q_x^{(\tau)} = \int_0^1 {}_s p_x^{(\tau)} \mu_{x+s}^{(\tau)} ds$$

or, more generally

$${}_t q_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \mu_{x+s}^{(\tau)} ds.$$

Force of a single decrement

- The force of decrement due to decrement j is defined as:

$$\mu_x^{(j)} = -\frac{1}{\ell_x^{(\tau)}} \frac{d}{dx} \ell_x^{(j)}.$$

- Notice that the denominator is NOT $\ell_x^{(j)}$ but is rather $\ell_x^{(\tau)}$.
- As a consequence, we see that

$$\mu_x^{(\tau)} = \sum_{j=1}^{m} \mu_x^{(j)}.$$

- The total force of decrement is (indeed) the sum of all the other partial forces of decrement.
- We can also show that

$$q_x^{(j)} = \int_0^1 {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds.$$

Illustrative exercise

Suppose that in a triple-decrement model, you are given constant forces of decrement, for a person now age x, as follows:

$$\begin{aligned}\mu_{x+t}^{(1)} &= b, \text{ for } t \geq 0,\\ \mu_{x+t}^{(2)} &= b, \text{ for } t \geq 0,\\ \mu_{x+t}^{(3)} &= 2b, \text{ for } t \geq 0.\end{aligned}$$

You are also given that the probability (x) will exit the group within 3 years due to decrement 1 is 0.00884.

Compute the length of time a person now age x is expected to remain in the triple decrement table.

Answer (to be discussed in lecture): $83\ 1/3$ years.

The associated single-decrement table (ASDT)

- For each of the causes of decrement in an MDT, a single-decrement table can be defined showing the operation of that decrement independent of the others.
  - called the associated single-decrement table (ASDT)
- Each table represents a group of lives reduced continuously by only one decrement. For example, a group subject only to death, but not to other decrements such as withdrawal.
- The associated probabilities in the ASDT are called absolute rates of decrements. For example, the absolute rate of decrement due to decrement j over the interval $(x, x+t]$ is ${}_tq_x'^{(j)}$.
- One should be able to explain intuitively why the following always hold true:

$$ {}_tq_x'^{(j)} \geq {}_tq_x^{(j)}. $$

Link between the MDT and the ASDT

- If given the absolute rates of decrements, say $q_x'^{(1)}, q_x'^{(2)}, \ldots, q_x'^{(m)}$, how do we derive the probabilities of decrements $q_x^{(1)}, q_x^{(2)}, \ldots, q_x^{(m)}$ in the MDT? And vice versa.
- The fundamental link: $\mu_x^{(j)} = \mu_x'^{(j)}$ for all $j = 1, 2, ..., m$.
- Therefore, it follows that

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)} \times {}_tp_x'^{(2)} \times \cdots \times {}_tp_x'^{(m)}. $$

- Furthermore, we note that

$$ \begin{aligned} {}_tq_x^{(j)} &= \int_0^t {}_sp_x^{(\tau)} \cdot \mu_{x+s}^{(j)} ds \\ &= \int_0^t {}_sp_x^{(\tau)} \cdot \mu_{x+s}'^{(j)} ds = \int_0^t \frac{{}_sp_x^{(\tau)}}{{}_sp_x'^{(j)}} {}_sp_x'^{(j)} \cdot \mu_{x+s}'^{(j)} ds. \end{aligned} $$

In the multiple decrement context

- We assume the following UDD assumption:

$$ {}_tq_x^{(j)} = t \cdot q_x^{(j)}, \quad \text{for } 0 \le t \le 1. $$

- This leads us to the following result:

$$ {}_tp_x'^{(j)} = \left(1 - t \cdot q_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}. $$

- Proof to be done in class.
- This result allows us to compute the absolute rates of decrements $q_x'^{(j)}$ given the probabilities of decrements in the multiple decrement model. In particular, when $t = 1$, we have

$$ q_x'^{(j)} = 1 - \left(1 - q_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}. $$

Illustrative example

In a double decrement table where cause d is death and cause w is withdrawal, you are given:

- both deaths and withdrawals are each uniformly distributed over each year of age in the double decrement table.
- $\ell_x^{(\tau)} = 1000$
- $q_x^{(w)} = 0.48$
- $d_x^{(d)} = 0.35 d_x^{(w)}$

Calculate $q_x'^{(d)}$ and $q_x'^{(w)}$.

Note: One way to check your results make sense is to ensure the inequality $q_x'^{(j)} \geq q_x^{(j)}$ is satisfied.

In the associated single decrement context

- We assume the following UDD assumption:

$$ {}_tq_x^{\prime(j)} = t \cdot q_x^{\prime(j)}, \quad \text{for } 0 \le t \le 1. $$

- This implies:

$$ {}_tp_x^{\prime(j)} \mu_{x+t}^{\prime(j)} = {}_tp_x^{\prime(j)} \mu_{x+t}^{(j)} = q_x^{\prime(j)}. $$

- Using the previous link, one can derive

$$ \begin{aligned} {}_tq_x^{(j)} &= \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds \\ &= \int_0^t \prod_{i \neq j} {}_sp_x^{\prime(i)} \, {}_sp_x^{\prime(j)} \mu_{x+s}^{\prime(j)} ds \\ &= q_x^{\prime(j)} \int_0^t \prod_{i \neq j} (1 - s \cdot q_x^{\prime(i)}) ds. \end{aligned} $$

- Use this integration to derive the probabilities of decrement given the absolute rates of decrements.

The case of two decrements

- When we have $m = 2$, we can derive

$$\begin{aligned} {}_tq_x^{(1)} &= q_x'^{(1)} \int_0^t \left(1 - s \cdot q_x'^{(2)}\right) ds \\ &= q_x'^{(1)} \left(t - \frac{1}{2} t^2 q_x'^{(2)}\right), \end{aligned}$$

  and similarly,

$${}_tq_x^{(2)} = q_x'^{(2)} \left(t - \frac{1}{2} t^2 q_x'^{(1)}\right).$$

- Check the case when $t = 1$.
- As an exercise, extend the derivation to the case of a triple decrement case.

Illustrative example 1

In a triple decrement table where each of the decrement in their associated single decrement tables satisfy the uniform distribution of decrement assumption, you are given:

- $q_x'^{(1)} = 0.03$ and $q_x'^{(2)} = 0.06$
- $\ell_x^{(\tau)} = 1,000,000$ and $\ell_{x+1}^{(\tau)} = 902,682$

Calculate $d_x^{(3)}$.

Illustrative example 2

In a triple decrement table, you are given that decrement (1) is death, decrement (2) is disability, and decrement (3) is withdrawal.

In addition, you have:

- $q'^{(1)}_{60} = 0.01$, $q'^{(2)}_{60} = 0.05$ and $q'^{(3)}_{60} = 0.10$.
- Withdrawals occur only at the end of the year.
- Mortality and disability are uniformly distributed over each year of age in the associated single decrement tables.

Calculate $q^{(3)}_{60}$.

Illustrative example 1

- An insurance policy issued to (50) will pay $\$40,000$ upon death if death is accidental and occurs within 25 years.
- An additional benefit of $\$10,000$ will be paid regardless of the time or cause of death.
- The force of accidental death at all ages is 0.01.
- The force of death for all other causes is 0.05 at all ages.
- You are given $\delta = 10\%$.
- Find the net single premium for this policy.
- [To be discussed in lecture.]

Illustrative example 2

- An employer provides his employees aged 62 the following one-year term benefits, payable at the end of the year of decrement:
  - $\$1$ if decrement results from cause 1;
  - $\$2$ if decrement results from cause 2; and
  - $\$6$ if decrement results from cause 3.
- Only three possible decrements exist.
- In their associated single-decrement tables, all three decrements follow de Moivre's Law with $\omega = 65$.
- You are given $i = 10\%$.
- Find the actuarial present value at age 62 of the benefits.

Asset share calculations

Asset shares refer to the projections of the assets expected to accumulate under a single policy (or a portfolio of policies). To illustrate, consider an insurance contract that pays:

- a benefit of $b_k^{(d)}$ at the end of year k for deaths during the year, and
- a benefit of $b_k^{(w)}$ at the end of year k for withdrawals of surrenders during the year.

The policy receives an annual contract premium of G at the beginning of the year.

It pays a percentage r_k of the premium for expenses plus a fixed amount of expense of e_k. Expenses occur at the beginning of the year.

- continued

In addition ,we have

- Interest rate is an effective annual rate of i.
- The probabilities of decrements are denoted by $q_{x+k-1}^{(d)}$ and $q_{x+k-1}^{(w)}$, respectively, for deaths and withdrawals.
- The probability of staying in force through year k is therefore

$$p_{x+k-1}^{(\tau)} = 1 - q_{x+k-1}^{(d)} - q_{x+k-1}^{(w)}.$$

Denote the asset share at the end of year k by AS_k with an initial asset share at time 0 of AS_0 which may or may not be zero.

For a new policy/contract, we may assume this is zero.

The recursion formula for asset shares

Beginning with $k = 1$, we find

$$[\mathsf{AS}_0 + G(1-r_1) - e_1](1+i) = b_1^{(d)} q_x^{(d)} + b_1^{(w)} q_x^{(w)} + \mathsf{AS}_1 \cdot p_x^{(\tau)},$$

and we get

$$\mathsf{AS}_1 = \frac{[\mathsf{AS}_0 + G(1-r_1) - e_1](1+i) - b_1^{(d)} q_x^{(d)} - b_1^{(w)} q_x^{(w)}}{p_x^{(\tau)}}.$$

This is easy to generalize as follows:

$$\mathsf{AS}_k = \frac{[\mathsf{AS}_{k-1} + G(1-r_k) - e_k](1+i) - b_k^{(d)} q_{x+k-1}^{(d)} - b_k^{(w)} q_{x+k-1}^{(w)}}{p_{x+k-1}^{(\tau)}}.$$

Do not memorize - use your intuition to develop the recursive formulas.

Illustrative example

For a portfolio of fully discrete whole life insurances of $1,000 on (30), you are given:

- the contract annual premium is $9.50;
- renewal expenses, payable at the start of the year, are 3% of premium plus a fixed amount of $2.50;
- $\text{AS}_{20} = 145$ is the asset share at the end of year 20;
- $\text{CV}_{21} = 100$ is the cash value payable upon withdrawal at the end of year 21;
- interest rate is $i = 7.5\%$ and the applicable decrement table is given below:

x	$q_x^{(d)}$	$q_x^{(w)}$
50	0.0062	0.0415
51	0.0065	0.0400

Calculate the asset share at the end of year 21.