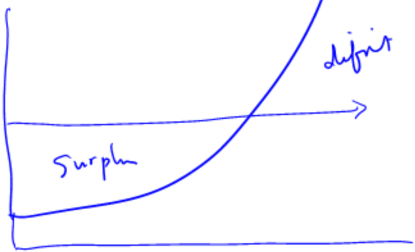
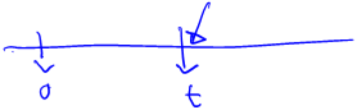
Reserves -

Policy Values

Lecture: Weeks 2-4



Chapter summary



- Insurance reserves (policy values)
  - what are they? how do we calculate them? why are they important?
- Reserves or policy values
  - benefit reserves (no expenses considered)
  - gross premium reserves (expenses accounted for)
  - prospective calculation of reserves (based on the future loss random variable)
  - retrospective calculation of reserves (not emphasized)
- Other topics to be covered (in separate slides)
  - analysis of profit or loss and analysis by source (mortality, interest, expenses)
  - asset shares
  - Thiele's differential equation for reserve calculation
  - policy alterations
- Chapters 7 (Dickson, et al.)

Mortality assumptions

For illustration purposes, we may base our calculations on the following assumptions:

- Illustrative Life Table (ILT)
  - the (official) Life Table used for Exam MLC with $i = 6\%$
- Standard Ultimate Survival Model, pp. 583, 586-587
  - introduced in Section 4.3
  - Makeham's law $\mu_x = A + Bc^x$, with $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$, and interest rate $i = 5\%$
- Standard Select Survival Model, pp. 583, 584-585
  - introduced in Example 3.13
  - the ultimate part follows the same Makeham's law as above; the select part follows

$$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}, \quad \text{for } 0 \le s \le 2,$$

    and interest rate $i = 5\%$

Insurance reserves (policy values)

100

- Money set aside to be able to cover insurer's future financial obligations as promised through the insurance contract.
  - reserves show up as a liability item in the balance sheet;
  - increases in reserves are an expense item in the income statement.
- Reserve calculations may vary because of:
  - purpose of reserve valuation: statutory (solvency), GAAP (realistic, shareholders/investors), mergers/acquisitions
  - assumptions and basis (mortality, interest) - may be prescribed
- Actuary is responsible for preparing an Actuarial Opinion and Memorandum: that the company's assets are sufficient to back reserves.
- Reserves are more often called provisions in Europe.
  - another term used is policy values

Why hold reserves?

- For several life insurance contracts:
  - the expected cost of paying the benefits generally increases over the contract term; but
  - the periodic premiums used to fund these benefits are level.
- The portion of the premiums not required to pay expected cost in the early years are therefore set aside (or provisioned) to fund the expected shortfall in the later years of the contract.
- Reserves also help reduce cost of benefits as they also earn interest while being set aside.
- Although reserves are usually held on a per-contract basis, it is still the overall responsibility of the actuary to ensure that in the aggregate, the company's assets are enough to back these reserves.

The insurer's future loss random variable

- At any future time $t \geq 0$, define the insurer's (net) future loss random variable to be

$$L_t^n = \mathsf{PVFB}_t - \mathsf{PVFP}_t.$$

prospective loss
$t = 0 \Rightarrow$ issue.

- For most types of policies, it is generally true that for $t \geq 0$, $L_t^n \geq 0$, i.e. $\mathsf{PVFB}_t \geq \mathsf{PVFP}_t$.
- If we include expenses, the insurer's (gross) future loss random variable is said to be

$$L_t^g = \mathsf{PVFB}_t + \mathsf{PVFE}_t - \mathsf{PVFP}_t.$$

- For our purposes, we define the expected value of this future loss random variable to be the reserve or policy value at time t:

Remark

$$_tV^n = \mathsf{E}\big[L_t^n\big] = \underbrace{\mathsf{E}\big[\mathsf{PVFB}_t\big]}_{\mathsf{APVFB}} - \underbrace{\mathsf{E}\big[\mathsf{PVFP}_t\big]}_{\mathsf{APVFP}}$$

or in the case with expenses,

$_0V = 0$.

$$_tV^g = \mathsf{E}\big[L_t^g\big] = \mathsf{E}\big[\mathsf{PVFB}_t\big] + \underbrace{\mathsf{E}\big[\mathsf{PVFE}_t\big]}_{\mathsf{APVFE}} - \mathsf{E}\big[\mathsf{PVFP}_t\big]$$

fully discrete whole life policy to (x) $\quad$ B = benefit

P = premium

based on EP

ignore expenses

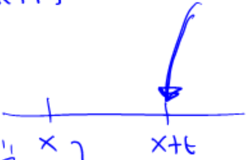
$$L_0^n = L_0 = Bv^{K+1} - P\ddot{a}_{\overline{K+1|}}$$

$$K = K_x$$

$$\downarrow$$

$$L_t = Bv^{K+1} - P\ddot{a}_{\overline{K+1|}}$$

$$K = K_{x+t}$$



$$E[L_t] = B\,\underbrace{E[v^{K+1}]}_{A_{x+t}} - P\,\underbrace{E[\ddot{a}_{\overline{K+1|}}]}_{\ddot{a}_{x+t}}$$

P + "riskmargin"

$$_tV = BA_{x+t} - P\ddot{a}_{x+t}$$

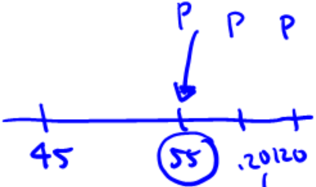
fully discrete whole life to (45)

$B = 1000$

Mortality ILT @ $i = 6\%$

Calculate at time $t=10$

ignore expenses



$$_{10}V^n = APVFB_{10} - APVFP_{10}$$

$$= \underbrace{1000 \underset{.30514}{A_{55}} - \underset{14.25727}{P} \; \underset{12.2758}{\ddot{a}_{55}}}_{}$$

$$= 130.1206$$

$$P = \frac{1000 \overset{.20120}{A_{45}}}{\underset{14.1121}{\ddot{a}_{45}}}$$

$$= 14.25727$$

Same policy but now with expenses

1st year 20% of G

renewal year 2% of G

Recalculate: $\underbrace{G\ddot{a}_{45}}_{\text{gross}} = 1000A_{45} + \underline{.18G} + \underline{.02G\ddot{a}_{45}}$

$\cancel{(.98\ddot{a}_{45} - .18)}G = \dfrac{1000A_{45}}{.98\ddot{a}_{45} - .18} = \underline{14.79008}$

Calculate $_{10}V^{g} = APVFB_{10} + APVFE_{10} - APVFG_{10}$

$= 1000A_{55} + .02G\ddot{a}_{55} - G\ddot{a}_{55}$

$127.8127 \Leftarrow 1000\underbrace{A_{55}}_{.30514} - .98\underbrace{G}_{14.79008}\underbrace{\ddot{a}_{55}}_{12.2758}$

$\underbrace{{}_{10}V^{n} = 130.1206}$ vs $\underbrace{{}_{10}V^{g} = 127.8127}$

$\Downarrow$

solvency

mat

conservation

↳ does not allow for expenses

new policies

= surplus strain

Some remarks I

$\underbrace{1000 - 103}_{\text{net amount at risk}}$

- ${}_tV^n$ and ${}_tV^g$ are respectively called net premium reserve and gross premium reserve. The primary difference between the two is the consideration of expenses.
- For Exam MLC, the term benefit reserve is often the preferred terminology to refer to the net premium reserve (no expenses).
- So if no confusion arises, we will often drop n and g in the superscripts for either the future loss random variable L_t or the reserve ${}_tV$.
- Note that $\mathrm{E}\big[L_t\big]$ is actually conditional on the survival of (x) at time t. Because otherwise, there is no reason to hold reserves when policy has been paid out (or matured or voluntarily withdrawn).
- Reserves are indeed released from the balance sheet when policy is paid out (or matured or voluntarily withdrawn).

Some remarks II

- Technically speaking, ${}_tV$ is to be the (smallest) amount for which the insurer is required to hold to be able to cover future obligations.
- We can see this from the following equations (here, we consider expenses, but if we ignore expenses, the term with expenses will simply be zero - same principle will hold):

  $$ {}_tV = \mathsf{APV}(\mathsf{FB}_t) + \mathsf{APV}(\mathsf{FE}_t) - \mathsf{APV}(\mathsf{FP}_t) $$

  Rewriting this, we get

  $$ \mathsf{APV}(\mathsf{FB}_t) + \mathsf{APV}(\mathsf{FE}_t) = \mathsf{APV}(\mathsf{FP}_t) + {}_tV. $$

- This equation tells us that the reserve ${}_tV$ is the balancing term in the equation to cover the deficiency of future premiums that arises at time t to cover future obligations (benefits plus expenses, if any).

A numerical illustration

Consider a whole life policy issued to a select age [40] with:

- \$100 of death benefit payable at the moment of death;
- premiums are annual payable at the beginning of each year;
- mortality follows the `Standard Select Survival Model` with $i = 5\%$; and
- mortality between integral ages follows the Uniform Distribution of Death (UDD).

The first step in reserve calculation is to determine the annual premiums. Let P be the annual premium in this case so that one can easily verify that

$\bar{A}_x = \frac{i}{\delta} A_x$

$$\begin{aligned} P &= 100 \times \frac{\bar{A}_{[40]}}{\ddot{a}_{[40]}} = 100 \times \frac{i}{\delta} \frac{A_{[40]}}{\ddot{a}_{[40]}} \\ &= 100 \left(\frac{0.05}{\log(1.05)} \right) \left(\frac{0.1209733}{18.45956} \right) = 0.6715928. \end{aligned}$$

A numerical illustration - continued

$[40] \longrightarrow [40]+5 = 45.$

The benefit reserve (or policy value) at the end of year 5 is given by

$$
\begin{aligned}
{}_5V &= \text{APV}(\text{FB}_5) - \text{APV}(\text{FP}_5) = 100 \times (i/\delta) A_{45} - P \times \ddot{a}_{45} \\
&= 100 \times \left(\frac{0.05}{\log(1.05)}\right)(0.151609) - 0.6715928 \times 17.81621 \\
&= 3.571607
\end{aligned}
$$

$0 \longrightarrow 3.57$

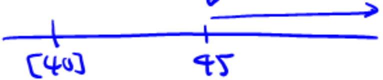
100

pricing P
valuation assumption

Note that we have calculated the policy value above as the expectation of a future loss random variable. We can also view reserve in terms of the insurer's account value after policies have been in force after 5 years (retrospectively).

Suppose that insurer issues N such similar but independent policies.

What happens to the insurer's account value after 5 years? [Done in lecture!]



Consider the same policy as in the previous example, but valued using the following assumptions:

age 40, but ignore select

Mortality is ILT

interest $i = 6\%$

$i = 6\%$ ILT

$$P = 100\,\frac{\bar{A}_{40}}{\ddot{a}_{40}} = 100\,\frac{i}{\delta}\,\frac{A_{40}}{\ddot{a}_{40}} = 1.121125$$

.16132

14.8166

$$_5V = 100\,\bar{A}_{45} - P\,\ddot{a}_{45}$$

$$= 100\,\frac{i}{\delta}\,A_{45} - P\,\ddot{a}_{45}$$

.20120

14.1121

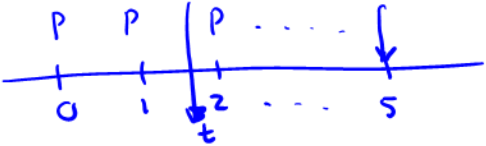
$$= \underline{\underline{4.896311}}$$

Retrospectively

N policies

P = premium per policy

i=6%



$$P*N\Big[(1.06)^5 + p_{40}(1.06)^4 + p_{40}\,p_{41}(1.06)^3$$

$$+ \cdots\cdots\Big]$$

$$(1.06)^5 v^t$$

$$- 100*N\int_0^5 {}_tp_{40}\,\mu_{40+t}(1.0\ \ )^{5-t}\,dt$$

$$= P*N*(1.06)^5\,\ddot{a}_{40:\overline{5}|} - 100*N\,(1.06)^5\,\bar{A}^{1}_{40:\overline{5}|}$$

$$= P * N * 1.06^5 \, \ddot{a}_{40:\overline{5}|} - 100 * N * 1.06^5 \, \bar{A}^{1}_{40:\overline{5}|}$$

ILT ✓

$P \rightarrow 1.121125$

$$\ddot{a}_{40:\overline{5}|} \rightarrow \ddot{a}_{40} - {}_5E_{40} \, \ddot{a}_{55}$$

$\ddot{a}_{40} = 14.8166$, ${}_5E_{40} = .73529$, $\ddot{a}_{55} = 14.124$

$$\bar{A}^{1}_{40:\overline{5}|} = \frac{i}{\delta} A^{1}_{40:\overline{5}|} \rightarrow A_{40} - {}_5E_{40} A_{45}$$

$$= 4.818103 * N$$

$$\frac{l_{45}}{l_{40}} * N = 4.896501$$

Verify the following calculations used in the last slide:

$$\underset{14.8166}{\ddot{a}_{40:\overline{5}|}} = \ddot{a}_{40} - \underset{.73529}{{}_5E_{40}}\, \overset{14.1121}{\ddot{a}_{45}} = 4.440255$$

$$\frac{i}{\delta} A^{1}_{40:\overline{5}|} = \frac{i}{\delta}\left(\underset{.16132}{A_{40}} - {}_5E_{40}\, \underset{.20120}{A_{45}}\right) = .01377714$$

$i = .06$

$\delta = \log(1.06)$

$l_{40} = 9313166$

$l_{45} = 9164051$

Summary

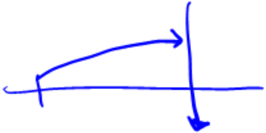
A reserve is calculated often prospectively

future values

but it is also equivalent to "retrospective" calculation

past values

Reserve = Accumulation of Premiums minus benefits



pp. 183 – 184 **book**

fully discrete whole life of B to (x)

at issue $\quad L_0 = Bv^{K+1} - P\ddot{a}_{\overline{K+1}|}$

$K = K_x \quad E[L_0] = 0 \Rightarrow P = B\dfrac{A_x}{\ddot{a}_x}$.

at time $t \quad L_t = \overbrace{B}^{\sim} v^{K+1} - P\ddot{a}_{\overline{K+1}|}$

$K = K_{x+t}$, $\quad$ conditional $T_x \geq t$

$E[L_t^n \mid T_x \geq t] = E[L_t^n] = {}_tV^n = \underbrace{BA_{x+t}}_{APV(FB_t)} - \underbrace{P\ddot{a}_{x+t}}_{APV(FP_t)}$

, $\quad L_t^n = Bv^{K+1} - P\left(\dfrac{1 - v^{K+1}}{d}\right) = \left(B + \dfrac{P}{d}\right)v^{K+1} - \dfrac{P}{d}$.

$$\mathrm{Var}[L_t^n] = \left(B + \frac{P}{d}\right)^2 \underbrace{\mathrm{Var}[v^{K+1}]}_{\left[{}^2A_{x+t} - (A_{x+t})^2\right]} \qquad K = K_{x+t}$$

$$ {}_tV^n = B A_{x+t} - P \ddot{a}_{x+t} \qquad P = \frac{B A_x}{\ddot{a}_x}$$

$$= B\left[A_{x+t} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t}\right] \qquad A_x = 1 - d\ddot{a}_x \qquad A_{x+t} = 1 - d\ddot{a}_{x+t}$$

$$= B\left[1 - \cancel{d\ddot{a}_{x+t}} - \frac{1 - \cancel{d\ddot{a}_x}}{\ddot{a}_x}\ddot{a}_{x+t}\right]$$

$$= B\left[1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right] = B\left[1 - \frac{P\ddot{a}_{x+t}}{P\ddot{a}_x}\right]$$

$$\ddot{a}_{x+t} = \frac{1-A_{x+t}}{d} \qquad \ddot{a}_x = \frac{1-A_x}{d}$$

$$= B\left[1 - \frac{(1-A_{x+t})/\cancel{d}}{(1-A_x)/\cancel{d}}\right] = \underbrace{B\left(\frac{A_{x+t}-A_x}{1-A_x}\right)}_{\text{intuition}}$$

Fully discrete reserves - whole life insurance

Consider the case of a fully discrete whole life insurance issued to a life (x) where premium of P is paid at the beginning of each year and benefit of $\$B$ is paid at the e.o.y. of death.

- The insurer's future loss random variable at time k (or at age $x+k$) is

  $$L_k = Bv^{K_{x+k}+1} - P\ddot{a}_{\overline{K_{x+k}+1}|},$$

  for $k = 0, 1, 2, \ldots$

- Applying the equivalence principle by solving $\mathsf{E}\left[L_0\right] = 0$, it can be verified that

  $$P = B \times \frac{A_x}{\ddot{a}_x} = B \times P_x.$$

- The benefit reserve (or policy value) at time k can be expressed as

  $${}_kV = \mathsf{E}\left[L_k\right] = B \times \left(A_{x+k} - P_x\, \ddot{a}_{x+k}\right).$$

- continued

The benefit reserve at time k is indeed equal to the difference between

$$\mathsf{APV}(\mathsf{FB}_k) = B \times A_{x+k}$$

and

$$\mathsf{APV}(\mathsf{FP}_k) = B \times P_x \, \ddot{a}_{x+k}$$

Sometimes, the variance is a helpful statistic and one can easily derive the variance of L_k with

$$\begin{aligned} \mathsf{Var}\left[L_k\right] &= \mathsf{Var}\left[B \cdot v^{K_{x+k}+1}\left(1 + \frac{P_x}{d}\right) - B \cdot \frac{P_x}{d}\right] \\ &= B^2 \times \left(1 + \frac{P_x}{d}\right)^2 \left[{}^2A_{x+k} - (A_{x+k})^2\right]. \end{aligned}$$

Other special formulas

Note that it can be shown that other special formulas for the benefit premium reserves for the fully discrete whole life hold:

- $_kV = 1 - d\ddot{a}_{x+k} - \left(\dfrac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+k} = 1 - \dfrac{\ddot{a}_{x+k}}{\ddot{a}_x}$

- $_kV = 1 - \dfrac{P_x + d}{P_{x+k} + d} = \dfrac{P_{x+k} - P_x}{P_{x+k} + d}$

- $_kV = 1 - \dfrac{1 - A_{x+k}}{1 - A_x} = \dfrac{A_{x+k} - A_x}{1 - A_x}$

- $P_x = A_x / \ddot{a}_x$
- $P_{x+t} = A_{x+t} / \ddot{a}_{x+t}$

Note that in these formulas we set $B = 1$. If the benefit amount B is not \$1, then simply multiply these formulas with the corresponding benefit amount.

A numerical illustration

Consider a fully discrete whole life policy of \$10,000 issued to a select age (40) with:

- mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$; and

One can verify that $P = 65.58717$ and the following table of benefit reserves:

$${}_1V = 10{,}000\left(1 - \frac{\overset{18.3403}{\ddot{a}_{41}}}{\underset{18.4578}{\ddot{a}_{40}}}\right)$$

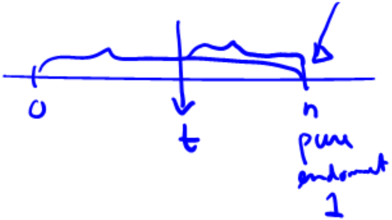
${}_0V = 0$ because of EP

$${}_kV = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$$

k	$\ddot{a}_{40+k}$	${}_kV$	k	$\ddot{a}_{40+k}$	${}_kV$
0	18.4578	0.000	13	16.4678	1078.103
1	18.3403	63.628	14	16.2676	1186.567
2	18.2176	130.096	15	16.0599	1299.123
3	18.0895	199.508	16	15.8444	1415.840
4	17.9558	271.966	17	15.6212	1536.774
5	17.8162	347.574	18	15.3901	1661.975
6	17.6706	426.437	19	15.1511	1791.478
7	17.5189	508.658	20	14.9041	1925.306
8	17.3607	594.340	21	14.6491	2063.467
9	17.1960	683.583	22	14.3861	2205.955
10	17.0245	776.487	23	14.1151	2352.744
11	16.8461	873.148	24	13.8363	2503.790
12	16.6606	973.658	25	13.5498	2659.027

$${}_{10}V = \underbrace{10{,}000\left(1 - \frac{16.8461}{17.0245}\right)}_{873.148}$$

Endowment policy of 1 - to (x)



$$P = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

$t \le n$

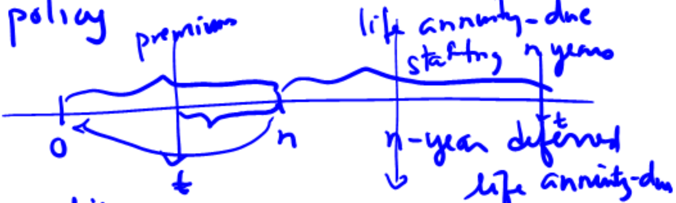
$$_tV^n = APV(FB_t) - APV(FP_t)$$

$$= A_{x+t:\overline{n-t}|} - P\,\ddot{a}_{x+t:\overline{n-t}|}$$

$$= 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{A_{x+t:\overline{n-t}|} - A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}$$

$_tV^n = 0,\ t > n$

life annuity policy to (x)



$1 of benefit

$$\underbrace{APV(FP_0)} = APV(FB_0)$$

$$P\ddot{a}_{x:\overline{n}|} = {}_{n|}\ddot{a}_x$$

$$= \underbrace{{}_nE_x\, \ddot{a}_{x+n}}$$

$$P = {}_{n|}\ddot{a}_x \,/\, \ddot{a}_{x:\overline{n}|}$$

P =

$t \le n$ $\quad {}_tV^n = APV(FB_t) - APV(FP_t)$

$$= {}_{n-t|}\ddot{a}_{x+t} - P\,\ddot{a}_{x+t:\overline{n-t}|}$$

$\underline{t > n}$ $\quad {}_tV^n = APV(FB_t) - \underbrace{APV(FP_t)}_{=0}$

$$= \ddot{a}_{x+t} - 0$$

Endowment policy

To simplify the formula development, assume $B = 1$.

- The future loss random variable at time $k \leq n$ (or at age $x + k$) is

$$L_k = v^{\min(K_{x+k}+1, n-k)} - P_{x:\overline{n|}}\, \ddot{a}_{\overline{\min(K_{x+k}+1, n-k)|}},$$

  for $k = 0, 1, \ldots, n$. Loss is zero for $k > n$.

- The benefit reserve at time k is

$${}_kV = A_{x+k:\overline{n-k|}} - P_{x:\overline{n|}}\, \ddot{a}_{x+k:\overline{n-k|}}.$$

- The variance of L_k is

$$\mathrm{Var}\big[L_k\big] = \left(1 + \frac{P_{x:\overline{n|}}}{d}\right)^2 \left[{}^2A_{x+k:\overline{n-k|}} - \left(A_{x+k:\overline{n-k|}}\right)^2\right].$$

Published SOA question #77

You are given:

$$\frac{A_x}{\ddot{a}_x}$$

- $P_x = 0.090$
- The benefit reserve at the end of year n for a fully discrete whole life insurance of \$1 on (x) is 0.563.
- $P_{x:\overline{n|}}^{\;\;\;1} = 0.00864$

$$\frac{{}_nE_x}{\ddot{a}_{x:\overline{n|}}}$$

Calculate $P^{1}_{x:\overline{n|}}$.

$$\frac{A^{1}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

$${}_nV = A_{x+n} - P_x\, \ddot{a}_{x+n} = .563$$

$$A_x = A^{1}_{x:\overline{n|}} + {}_nE_x\, A_{x+n}$$

$$A_{x+n} = \frac{A_x - A^{1}_{x:\overline{n|}}}{{}_nE_x}$$

$$\ddot{a}_{x+n} = \frac{\ddot{a}_x - \ddot{a}_{x:\overline{n|}}}{{}_nE_x}$$

$$ {}_nV_x = .563 = \frac{A_x - A_{x:\overline{n}|}^{\;1}}{{}_nE_x} - P_x \frac{\ddot{a}_x - \ddot{a}_{x:\overline{n}|}}{{}_nE_x} $$

$P_x = \dfrac{A_x}{\ddot{a}_x}$

$P_x \ddot{a}_x = A_x$

$\cancel{A_x - P_x \ddot{a}_x = 0}$

$$ = \frac{\left(\cancel{A_x} - \cancel{A_{x:\overline{n}|}^{\;1}} - \cancel{P_x \ddot{a}_x} + \cancel{P_x \ddot{a}_{x:\overline{n}|}}\right) / \ddot{a}_{x:\overline{n}|}}{{}_nE_x / \ddot{a}_{x:\overline{n}|}} $$

$$ = \frac{\overset{.09}{P_x} - P_{x:\overline{n}|}^{\;1}}{P_{x:\overline{n}|}^{\;\;1}} = .563 $$

$P_{x:\overline{n}|}^{\;\;1} \sim .00864$

$$ P_{x:\overline{n}|}^{\;1} = .09 - \frac{.563}{.00864} = .08513568 $$

Illustrative example 1

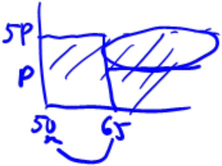
For a special fully discrete whole life insurance on (50), you are given:

- The death benefit is \$50,000 for the first 15 years and reduces to \$10,000 thereafter.
- The annual benefit premium is $5P$ for the first 15 years and reduces to P thereafter.
- Mortality follows the `Illustrative Life Table`.
- $i = 6\%$

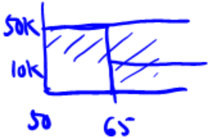
Calculate the following:

1. the value of P;
2. the ~~benefit~~ net premium reserve at the end of 10 years; and
3. the ~~benefit~~ net premium reserve at the end of 20 years.

$$APV(FP_0) = APV(FB_0)$$



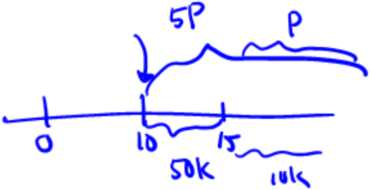
$$5P\,\ddot{a}_{50} - 4P\,{}_{15}E_{50}\,\ddot{a}_{65}$$



$$= 50000\,A_{50} - 40000\,{}_{15}E_{50}\,A_{65}$$

$$P = \frac{50000(.24905) - 40000(.342125)(.48782)}{5(\ddot{a}_{50}) - 4(.342125)\,\ddot{a}_{65}}$$

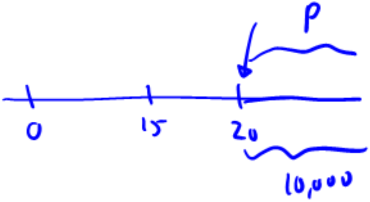
119.663



$${}_{10}V = \underbrace{APV(FB_{10})}_{} - APV(FP_{10})$$

$$= \left(50{,}000\, A_{60} - 40{,}000\; {}_5E_{60}\, A_{65}\right) - \left(5P\,\ddot{a}_{60} - 4P\; {}_5E_{60}\, \ddot{a}_{65}\right)$$

$$= \underline{\underline{3{,}949.575}}$$



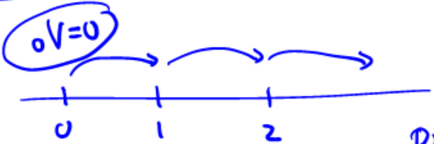
$${}_{20}V = 10{,}000\, A_{70} - P\,\ddot{a}_{70}$$

$$= \underline{4{,}285.892}$$

$${}_tV \overset{g}{=} APV(FB_t) + \underbrace{APV(FE_t)}_{} - APV(FP_t)$$

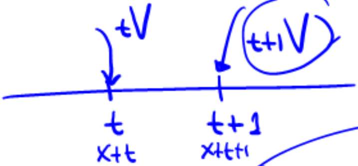
Recursive fully discrete whole life -

Equivalence Principle



product

$E[L_0]=0 \Rightarrow {}_0V = \emptyset$ $\$1, (x)$



$v^{k+1} - P\ddot{a}_{\overline{k+1}|}$

$P = \dfrac{A_x}{\ddot{a}_x}$

$${}_{t+1}V = E[L_{t+1}] = A_{x+t+1} - P\ddot{a}_{x+t+1}$$

$$(1 + vp_{x+t}\ddot{a}_{x+t+1})$$

$${}_tV = E[L_t] = A_{x+t} - P\ddot{a}_{x+t}$$

$$(vq_{x+t} + vp_{x+t}A_{x+t+1})$$

$$= v\, q_{x+t} - P + v\, p_{x+t} \left(\underbrace{A_{x+t+1} - P\, \ddot{a}_{x+t+1}}_{{}_{t+1}V} \right)$$

$$ {}_{t+1}V = \frac{({}_tV + P) - v\, q_{x+t}}{\cancel{v}\, p_{x+t}} \times (1+i)$$

$P > {}_tV$

t $\qquad$ $t+1$

$$ {}_{t+1}V = \frac{({}_tV + P)(1+i) - q_{x+t}}{1 - q_{x+t}}$$

If B_t = benefit,

$$ {}_{t+1}V = \frac{({}_tV + P)(1+i) - B_t \cdot q_{x+t}}{1 - q_{x+t}}$$

$${}_{t+1}V(1-q_{x+t}) = ({}_tV+P)(1+i) - B_t\, q_{x+t}$$

net premium reserve

$${}_{t+1}V = ({}_tV+P)(1+i) - (\underbrace{B_t - {}_{t+1}V}_{\text{net amount at risk}})\, q_{x+t}$$

net amount at risk

death strain

f = % of premium

e = fixed expenses

E = claim settlement expense

gross premium reserve

$$\boxed{{}_{t+1}V^g = ({}_tV^g + G(1-f) - e)(1+i) - (B_t - {}_{t+1}V^g + E)\, q_{x+t}}$$

Recursive formulas

To motivate development of recursive formulas, consider a fully discrete whole life insurance of \$B to (x). It can be shown (done in lecture) that:

$$_{k+1}V = \frac{\left({}_kV + P\right)(1+i) - Bq_{x+k}}{1 - q_{x+k}},$$

no expenses

with $k = 1, 2, \ldots$ and $_0V = 0$. One can verify the following calculations of the successive reserves for $B = 10,000$. See slides page 13.

$P = 68.58717$

$$_1V = \frac{68.58717(1.05) - 10{,}000\left(\frac{.52722}{1000}\right)}{1 - \frac{.52722}{1000}} = \underline{63.628}$$

k	$1000q_{40+k}$	$_kV$	k	$1000q_{40+k}$	$_kV$
0	0.52722	0.000	13	1.62346	1078.103
1	0.56531	63.628	14	1.79736	1186.567
2	0.60813	130.096	15	1.99278	1299.123
3	0.65625	199.508	16	2.21239	1415.840
4	0.71033	271.966	17	2.45917	1536.774
5	0.77112	347.574	18	2.73648	1661.975
6	0.83944	426.437	19	3.04808	1791.478
7	0.91622	508.658	20	3.39821	1925.306
8	1.00252	594.340	21	3.79161	2063.467
9	1.09952	683.583	22	4.23360	2205.955
10	1.20853	776.487	23	4.73017	2352.744
11	1.33104	873.148	24	5.28801	2503.790
12	1.46873	973.658	25	5.91465	2659.027

n-year term insurance

$$ {}_{t+1}V = \frac{({}_tV + P)(1+i) - B\,q_{x+t}}{1 - q_{x+t}} , \quad t < n-1 $$

$$ {}_tV = \phi , \quad t \geq n $$

n-year endowment insurance

$$ {}_{t+1}V = \frac{({}_tV + P)(1+i) - B\,q_{x+t}}{1 - q_{x+t}} $$

${}_nV = B$ = pure endowment at end of n years!

${}_tV = \phi$, $t > n$.

* useful for calculating P where B depends on ${}_tV$!

Gross premium reserve calculation

Consider a fully discrete whole life policy of \$10,000 issued to a select age (40) with:

- mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$; and

Suppose expenses consist of: (a) \$5 per 1,000 of death benefit in the first year and (b) \$2 per 1,000 of death benefit in subsequent years.

It can be shown that the gross annual premium, G, is

$$
\begin{aligned}
G &= \frac{10000A_{40} + 30 + 20\ddot{a}_{40}}{\ddot{a}_{40}} \\
&= \frac{10000(0.1210592) + 30 + 20(18.45776)}{18.45776} \\
&= 87.21251.
\end{aligned}
$$

- continued

To calculate gross premium reserves, use recursive formulas with ${}_0V = 0$:

$$ {}_1V = \frac{({}_0V + G - 50)(1.05) - 10000q_{40}}{1 - q_{40}}, \quad \text{and} $$

$$ {}_{k+1}V = \frac{({}_kV + G - 20)(1.05) - 10000q_{40+k}}{1 - q_{40+k}}, \quad \text{for } k = 1, 2, \ldots $$

k	$1000q_{40+k}$	${}_kV$	k	$1000q_{40+k}$	${}_kV$
0	0.52722	0.000	13	1.62346	1051.338
1	0.56531	33.819	14	1.79736	1160.127
2	0.60813	100.487	15	1.99278	1273.021
3	0.65625	170.106	16	2.21239	1390.087
4	0.71033	242.781	17	2.45917	1511.384
5	0.77112	318.617	18	2.73648	1636.961
6	0.83944	397.716	19	3.04808	1766.852
7	0.91622	480.184	20	3.39821	1901.082
8	1.00252	566.123	21	3.79161	2039.658
9	1.09952	655.634	22	4.23360	2182.573
10	1.20853	748.817	23	4.73017	2329.802
11	1.33104	845.768	24	5.28801	2481.301
12	1.46873	946.579	25	5.91465	2637.004

Compare these values with the benefit reserves. What do you observe?

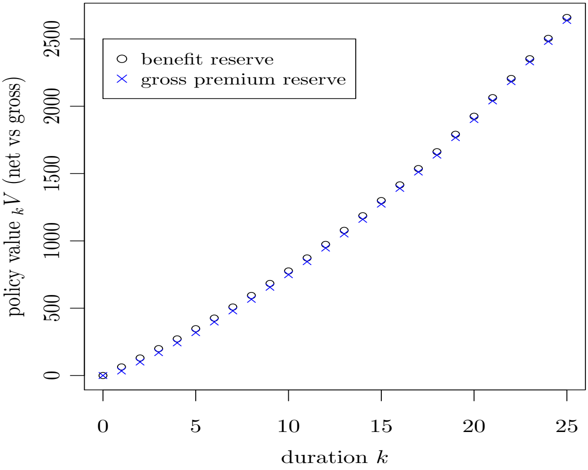


Figure: Comparison between benefit reserve and gross premium reserve

A generalization of recursive relations

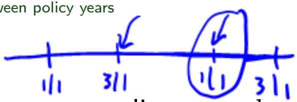
The reserve in the next period $t+1$ can be shown to be

$$ {}_{t+1}V = \frac{\big({}_tV + G_t - e_t\big)(1+i_t) - \big(B_{t+1} + E_{t+1}\big)q_{x+t}}{1 - q_{x+t}}. $$

Intuitively, we have:

- accumulate previous reserves plus premium (less expenses) with interest;
- deduct death benefits (plus any claims-related expenses) to be paid at the end of the year; and
- divide the reserves by the proportion of survivors.

Valuation between policy years



Sometimes we may want to compute reserves between policy years k and $k+1$, say at $k+h$ for some $0 < h < 1$.

One may use the recursive formula but with caution:

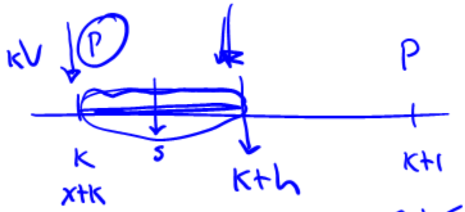
- timing of the premium payments and expenses (if any)
- timing of the payment of the death benefit

Consider the whole life policy considered in slides page 8.

The reserve at time $k+h$ can be derived (assuming UDD between integral ages):

$$
\begin{aligned}
{}_{k+h}V &= \frac{\left({}_kV + P\right)(1+i)^h - B \cdot \displaystyle\int_0^h (1+i)^{h-s} {}_sp_{[x]+k}\mu_{[x]+k+s} ds}{1 - \displaystyle\int_0^h {}_sp_{[x]+k}\mu_{[x]+k+s} ds} \\
&= \frac{\left({}_kV + P\right)(1+i)^h - B \cdot \frac{e^{\delta h}-1}{\delta} \cdot q_{[x]+k}}{1 - h \cdot q_{[x]+k}}
\end{aligned}
$$

[x]



$0 < h < 1$

$$
{}_{k+h}V = \frac{({}_kV+P)(1+i)^h - B\int_0^h (1+i)^{h-s}\,{}_sp_{x+k}\,\mu_{x+k+s}\,ds}{1-\underbrace{\int_0^h {}_sp_{x+k}\,\mu_{x+k+s}\,ds}_{\text{prop \# of death}}}
$$

UDD

$$
{}_sp_{x+k}\,\mu_{x+k+s} = s\cdot q_{x+k}
$$

$$
= \frac{({}_kV+P)(1+i)^h - B\,q_{x+k}\,(1+i)^h\int_0^h (1+i)^{-s}\cdot s\,ds}{1-q_{x+k}\underbrace{\int_0^h s\,ds}_{h}}
$$

$$\underbrace{\int_0^h s\,v^{s}\,ds}_{=\frac{e^{\delta h}-1}{\delta}} \qquad v^s = e^{-\delta s}$$

$$= \frac{({}_kV+P)(1+i)^n - B\,q_{x+k}\,\frac{e^{\delta h}-1}{\delta}}{1-h\cdot q_{x+k}}$$

Q#277 fully discrete whole life to (x)

linearly interpolate

$$\frac{({}_3V+P)+{}_4V}{2}$$

UDD $q_{x+3}=.101$

$_3V = 96$

$b = 360$

$i = 6\%$

$P = 24$

Calculate $_{3.5}V = ?$

$${}_{3.5}V = \frac{({}_3V+P)(1+i)^{1/2} - b\cdot {}_{1/2}q_{x+3}\cdot v^{1/2}}{1-{}_{1/2}q_{x+3}}$$

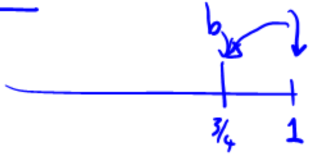
$$= \frac{(96+24)(1.06)^{1/2} - 360\cdot\frac{1}{2}(.101)(1.06)^{-1/2}}{1-\frac{1}{2}(.101)}$$

$$= \underline{\underline{111.5214}}$$

$$\frac{({}_3V+P)(1+i) - b\,q_{x+3}}{1-q_{x+3}},$$

compare

$${}_{3.5}V = \frac{({}_3V+P)+{}_4V}{2}$$



$${}_{3.75}V = \frac{({}_3V+P)(1+i)^{3/4} - b\cdot\frac{3}{4}q_{x+3}\,v^{1/4}}{1-\frac{3}{4}\cdot q_{x+3}},$$

net amount at risk

$\underbrace{\qquad\qquad}$

$B - {}_kV$

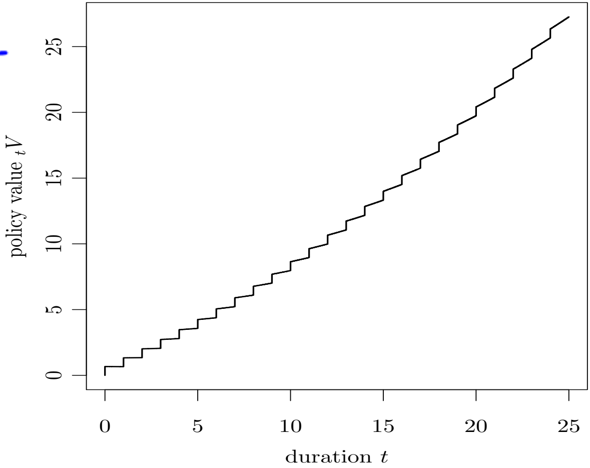


Figure: An illustration of the value of (benefit) reserves between policy years

Illustrative example 2

$APV(FP_0) = APV(FB_0)$

reserve

For a special single premium 20-year term insurance on (70):

- The death benefit, payable at the end of the year of death, is equal to 1000 plus the benefit reserve.
- $q_{70+k} = 0.03$, for $k = 0, 1, 2, \ldots$
- $i = 0.07$

${}_0V = 0$ ${}_{20}V = 0$

P x x x x .. 20

Calculate the single benefit premium for this insurance.

Best strategy is to recursive equation

20 year term $\quad {_{20}V} = 0 \quad$ $q = .03$ always

$$ {_1V} = \frac{(\cancel{_0V}^{\,0} + P)(1.07) - (1000 + {_1V})(.03)}{1 - .03} $$

$$ {_1V}(1 - \cancel{.03}) = P(1.07) - (1000 + \cancel{_1V})(.03) $$

$$ {_1V} = P(1.07) - 1000(.03) $$

$$ {_2V} = \frac{[P(1.07) - 1000(.03)](1.07) - (1000 + \cancel{_2V})(.03)}{\cancel{1 - .03}} $$

$$ {_2V}(1 - \cancel{.03}) $$

$$ {_2V} = P(1.07)^2 - 1000(.03)(1.07 + 1) $$

$$ \vdots $$

$$ 0 = {_{20}V} = \underbrace{P(1.07)^{20} - \overbrace{1000(.03)}^{30}\left[\overbrace{1.07^{20-1} + 1.07^{20-2} + \dots + 1}\right]}_{=0} $$

$$P = \frac{30\left(\bar{s}_{\overline{20|}}\right)}{(1.07)^{20}} \qquad \frac{1.07^{20}-1}{.07}$$

$$= 317.8204$$

<u>Variation</u>:

① P is yearly, not just single

② endowment, nV = pure endowment!

Net amount at risk

- The difference $B_{t+1} + E_{t+1} - {}_{t+1}V$ is called the net amount at risk.
- Sometimes called death strain at risk (DSAR) or sum at risk.
- The recursive formula can then alternatively be written as

$$\big({}_tV + G_t - e_t\big)(1 + i_t) = {}_{t+1}V + \big(B_{t+1} + E_{t+1} - {}_{t+1}V\big)q_{x+t}$$

where the term $\big(B_{t+1} + E_{t+1} - {}_{t+1}V\big)q_{x+t}$ can then be called the expected net amount at risk.

Published SOA question #118

For a special fully discrete three-year term insurance on (x):

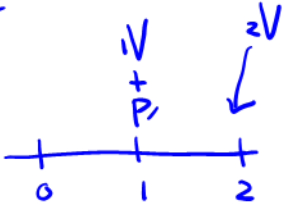
P

- Level benefit premiums are paid at the beginning of each year.
- Benefit amounts with corresponding death probabilities are

k	b_{k+1}	q_{x+k}
0	200,000	0.03
1	150,000	0.06
2	100,000	0.09

- $i = 0.06$

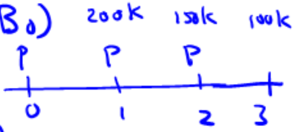
Calculate the initial benefit reserve for year 2.



${}_1V + P$

Calculate P

$$APV(FP_0) = APV(FB_0)$$



$$v = \frac{1}{1.06}$$

$$P + Pv(.97) + Pv^2(.97)(.94)$$

$$200{,}000\,v(.03) + 150{,}000\,v^2(.97)(.06) + 100{,}000\,v^3(.97)(.94)(.09)$$

$$P(\;\ldots\;\ldots\;) = \quad \Rightarrow \boxed{P = 7452.572}$$

$${}_0V = 0 \text{ by E.P.}$$

$${}_1V = \frac{(0+P)(1.06) - 200{,}000(.03)}{1 - .03} = 1958.48$$

initial reserve in year 2 $= {}_1V + P = \underline{\underline{9411.052}}$ ✓

SOA MLC Fall 2014 question #13

B

For a fully discrete whole life insurance of 100,000 on (45), you are given:

- The gross premium reserve at duration 5 is 5500 and at duration 6 is 7100.
- $q_{50} = 0.009$
- $i = 0.05$
- Renewal expenses at the start of each year are 50 plus 4% of the gross premium.
- Claim expenses are 200.

$_5V = 5500$

$_6V = 7100$

Calculate the gross premium.

$$cV = \frac{\left(5V + G - 50 - 4\%\,G\right)(1.05) - \left(\overset{+200}{100{,}000}\right)(.009)}{1 - .009}$$

$$G(\cancel{.96}) = \frac{\overset{+7100}{cV}(1-.009) - \overset{5500}{5V}(1.05) + 50(1.05) + 100{,}200(.009)}{.96}$$

$$= \underline{\underline{2197.817}} \approx 2200$$

---

4 $\Rightarrow$ <u>recursive</u> $\qquad \underline{0,\ 1,\ 2}$

Fully continuous reserves - whole life

Consider now the case of a fully continuous whole life insurance with an annual premium rate of $\bar{P}(\bar{A}_x)$. – P or $\bar{P}$.

- The future loss random variable at time t (or at age $x+t$):

$T = T_{x+t}$

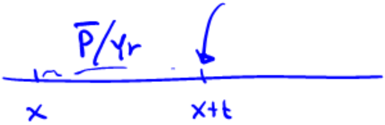
$$L_t = v^{T_{x+t}} - \bar{P}(\bar{A}_x)\, \overbrace{\bar{a}_{\overline{T_{x+t}}|}}^{\frac{1-v^{T_{x+t}}}{\delta}} = v^{T_{x+t}}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}.$$

- The benefit reserve at time t is

$${}_tV = \mathsf{E}\left[L_t\right] = \overbrace{\bar{A}_{x+t}}^{\text{APV(FB)}} - \overbrace{\bar{P}(\bar{A}_x)\,\bar{a}_{x+t}}^{\text{APV(FP)}}.$$

- The variance of L_t is

$$\mathsf{Var}\left[L_t\right] = \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 \left[{}^{\overset{2\delta}{2}}\bar{A}_{x+t} - \left(\bar{A}_{x+t}\right)^2\right].$$



$$_tV = APVFB - APVFP$$

$$= \bar{A}_{x+t} - \bar{P} \cdot \bar{a}_{x+t}$$

$$\bar{P} = \frac{\bar{A}_x}{\bar{a}_x}, \qquad \bar{A}_x = 1 - \delta \bar{a}_x$$

$$\bar{A}_x = 1 - \delta \bar{a}_x$$

$$\bar{A}_{x+t} = 1 - \delta \bar{a}_{x+t}$$

$$1 - \delta \bar{a}_{x+t} - \left(\frac{1 - \delta \bar{a}_x}{\bar{a}_x} \right) \bar{a}_{x+t} = \underbrace{1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}}$$

$$1 - \frac{\bar{P} \bar{a}_{x+t}}{\bar{P} \bar{a}_x}$$

$$_kV = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$$

Constant force

$T_x \sim \mathrm{Exp}(\mu)$

Recall: $\bar{A}_x = \dfrac{\mu}{\mu+\delta}$

$\bar{a}_x = \dfrac{1}{\mu+\delta}$

$\bar{P} = \mu$

${}_tV = 0$

$$ {}_tV = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} $$

$$ = 1 - \frac{1/\cancel{\mu+\delta}}{1/\cancel{\mu+\delta}} = 1 - 1 = 0 $$

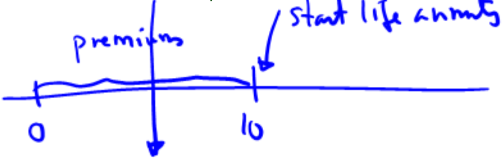
De Moivre's

Other formulas

Some continuous analogues of the discrete case:

- ${}_tV = 1 - \dfrac{\bar{a}_{x+t}}{\bar{a}_x}$ in terms of annuities
- ${}_tV = \dfrac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}$ in terms of premiums
- ${}_tV = \dfrac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}$ in term of insurance

Illustrative example 3



For a 10-year deferred whole life annuity of 1 on (35) payable continuously, you are given:

- Mortality follows deMoivre's law with $\omega = 85$.
- Level benefit premiums are payable continuously for 10 years.
- $i = 0$

Calculate the benefit reserve at the end of five years.

EP: $APVFP_0 = APVFB_0$

$$\bar{P}\,\bar{a}_{35:\overline{10|}} = 1 \cdot {}_{10}E_{35}\,\bar{a}_{45}$$

$$\frac{1}{2}\frac{t^2}{40}\Big|_0^{40}$$

$$\bar{a}_{45} = \int_0^{40} \cancel{v^t}\, {}_tp_{45}\,dt = \int_0^{40} (1 - t/40)\,dt$$

$$= 40 - 20 = \underline{\underline{20}}$$

$\omega = 85$

$T_0 \sim (0, 85)$

$T_x \sim (0, 85 - x)$

$T_{45} \sim (0, 40)$

$v = 1$

$${}_tp_{45} = \underbrace{P(T_{45} > t)} = 1 - t/40$$

$$\int_t^{40} \frac{1}{40}\,dt = 1 - t/40$$

$T_{35} \sim (0, 50)$

$$\bar{a}_{35:\overline{10}|} = \int_0^{10} \cancel{v^t}\, \underset{(1 - t/50)\,dt}{{}_tp_{35}}\,dt = 10 - \frac{1}{2}\,\frac{100}{50} = 9$$

$v = 1$

$${}_{10}E_{35} = \cancel{v^{10}}\, {}_{10}p_{\underline{\underline{35}}} = \frac{40}{50} = \frac{4}{5} \Rightarrow \bar{P} = \frac{\frac{4}{5}(\overset{4}{\cancel{20}})}{9} = \frac{16}{9}$$

$${}_5V = APVFB_5 - APVFP_5 = {}_5E_{40}\,\bar{a}_{45} - \bar{P}\,\bar{a}_{40:\overline{5}|}$$

$${}_5V = \cancel{40}\,\underbrace{{}_5p_{40}}\,(20) - \frac{16}{9}\int_0^5 \left(1 - t/45\right) dt$$

$$\left(1 - \frac{5}{45}\right)(20) - \frac{16}{9}\left(5 - \frac{1}{2}\frac{(25)}{45}\right) = \frac{760}{81}$$

0

Illustrative example 4 - modified SOA MLC Spring 2012

A special fully discrete 3-year endowment insurance on (x) pays death benefits as follows:

Year of Death	Death Benefit
1	\$ 10,000
2	\$ 20,000
3	\$ 30,000

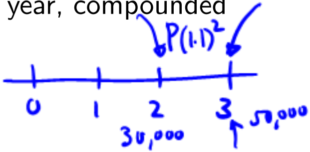
P

$P(1.1)$

$P(1.1)^2$

You are given:

- The endowment benefit amount is \$ 50,000
- Annual benefit premiums increase at 10% per year, compounded annually.
- $i = 0.05$
- $q_x = 0.08 \qquad q_{x+1} = 0.10 \qquad q_{x+2} = 0.12$



Calculate the benefit reserve at the end of year 2.

$${}_2V = \underbrace{APVFB_2} - \underbrace{APVFP_2}$$

$$= \left(30{,}000\, v\, q_{x+2} + 50{,}000\, v\, p_{x+2}\right) - P(1.1)^2$$

First calculate premium!

$$v = \frac{1}{1.05}$$

$$\underbrace{APVFB_0} = \underbrace{APVFP_0}$$

$$\underbrace{10{,}000\, v\, \overset{.08}{q_x} + 20{,}000\, v^2\, \overset{.92}{p_x}\, \overset{.10}{q_{x+1}} + 30{,}000\, v^3\, \overset{.92}{p_x}\, \overset{.9}{p_{x+1}}\, \overset{.12}{q_{x+1}} + 50{,}000\, v^3\, \underset{.92}{p_x}\, \underset{.9}{p_{x+1}}\, \underset{.88}{p_{x+1}}}$$

$$\underbrace{P + P(1.1)\, p_x + P(1.1)^2\, p_x\, p_{x+1}}$$

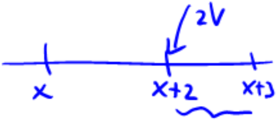
$$P\underbrace{\left[1 + 1.1\, \underset{.92}{p_x} + (1.1)^2\, \underset{.92}{p_x}\, \underset{.90}{p_{x+1}}\right]}$$

$$\Rightarrow \quad P = \frac{36477.10}{2.872544} = 12{,}698.53$$

You should verify the following calculations:

Note that reserve is conditional on (x) reaching age x+2

& at year 2, only 1 year left



$${}_2V = \underbrace{\left(30{,}000 \, \frac{1}{1.05} \, 0.12 + 50{,}000 \, \frac{1}{1.05} \, 0.88\right) - \underbrace{12{,}698.53}_{P}\,(1.1)^2}_{= 29{,}968.11}$$

I guess the reserve has to be slightly lower than 30k because of still yet the premium to be collected at beg of year

One can also show the reserves recursively

$${}_0V = \phi$$

$${}_1V = \frac{P(1.05) - 10{,}000(.08)}{1 - .08} = 13623.33$$

$${}_2V = \frac{(13623.33 + P(1.1))(1.05) - 20{,}000(.10)}{1 - .10} = 29968.11$$

finally,

$${}_3V = \frac{(29968.11 + P(1.1)^2)(1.05) - 30{,}000(.12)}{1 - .12} = \underline{\underline{50{,}000}}$$

$\Downarrow$

equal to the endowment benefit!

Note you start ${}_0V = 0$ only if Premiums are determined by equivalence principle!

Other terminologies and notations used

Expression	Other terms/symbols used
reserves	policy values
future loss random variable	prospective loss
net amount at risk	death strain at risk (DSAR) sum at risk
reserve at end of the year	terminal reserve
reserve at beginning of year plus applicable premium	initial reserve