Exercise 6.19

(a) Let $K = K_{[30]}$ be the curtate future lifetime of a select age 30 and P be the gross annual premium. The loss-at-issue random variable is

$$L_0 = \text{PVFB}_0 + \text{PVFE}_0 - \text{PVFP}_0,$$

where

$$\text{PVFB}_0 = 250000 \times \begin{cases} v^{K+1}(1.025)^K, & \text{for } K < 25 \\ v^{25}(1.025)^{25}, & \text{for } K \geq 25 \end{cases} = 250000 Z_1$$

and

$$\begin{aligned} \text{PVFE}_0 &= 1200 + 0.39P + 0.01P \times \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{for } K < 25 \\ \ddot{a}_{\overline{25}|}, & \text{for } K \geq 25 \end{cases} \\ &= 1200 + 0.39P + 0.01\frac{P}{d} - 0.01\frac{P}{d} \times \begin{cases} v^{K+1}, & \text{for } K < 25 \\ v^{25}, & \text{for } K \geq 25 \end{cases} \\ &= 1200 + 0.39 + 0.01\frac{P}{d} - 0.01\frac{P}{d} Z_2 \end{aligned}$$

and

$$\begin{aligned} \text{PVFE}_0 &= P \times \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{for } K < 25 \\ \ddot{a}_{\overline{25}|}, & \text{for } K \geq 25 \end{cases} \\ &= \frac{P}{d} - \frac{P}{d} \times \begin{cases} v^{K+1}, & \text{for } K < 25 \\ v^{25}, & \text{for } K \geq 25 \end{cases} \\ &= \frac{P}{d} - \frac{P}{d} Z_2 \end{aligned}$$

The desired results immediately follows:

$$\begin{aligned} L_0 &= 250000 Z_1 + 1200 + 0.39P - 0.99\frac{P}{d} + 0.99\frac{P}{d} Z_2 \\ & \quad \text{or after some re-arrangement} \\ &= 250000 Z_1 + 0.99\frac{P}{d} Z_2 + 1200 + 0.39P - 0.99\frac{P}{d} \end{aligned}$$

(b) Based on the equivalence principle, we set $\text{E}[L_0] = 0$ to solve for P:

$$P = \frac{250000\text{E}\big[Z_1\big] + 1200}{(0.99/d)\big(1 - \text{E}\big[Z_2\big]\big) - 0.39},$$

where

$$\text{E}\big[Z_1\big] = \frac{1}{1.025} A^{\ 1}_{[30]:\overline{25}|\, i_1} + {}_{25}E_{[30]\, i_1},$$

with $A^{\;1}_{[30]:\overline{25}|\,i_1}$ and ${}_{25}E_{[30]\,i_1}$ being, respectively, 25-year term insurance and pure endowment evaluated at interest rate

$$i_1 = \frac{1.05}{1.025} - 1,$$

and

$$\mathrm{E}\big[Z_2\big] = A_{[30]:\overline{25}|},$$

a 25-year endowment insurance evaluated at interest rate i. Based on the results of the R code below for calculating these actuarial values, we get

$$\mathrm{E}\big[Z_1\big] = \frac{1}{1.025}(0.01270741) + 0.5371817 = 0.5495792,$$

and

$$\mathrm{E}\big[Z_2\big] = A^{\;1}_{[30]:\overline{25}|} + {}_{25}E_{[30]} = 0.008766631 + 0.2897508 = 0.2985174.$$

Thus, we have

$$P = \frac{250000(0.5495792) + 1200}{(0.99/(1-1.05^{-1}))(1-0.2985174) - 0.39} = 9764.444.$$

```
# Makeham parameters:
A <- .00022
B <- 2.7*10^(-6)
c <- 1.124
# 25-year term to [30] calculated at interest rate i1 = (1.05/1.025)-1
x <- 30:131
kp30 <- rep(0,length(x))
temp1 <- ((1-.9)/log(0.9))*A
temp2 <- ((c-0.9)/log(0.9/c))*B*c^(30)
kp30[1] <- exp(0.9*(temp1+temp2))
kp30[2] <- exp(((1-.9^2)/log(0.9))*A + ((c^2 - 0.9^2)/log(0.9/c))*B*c^(30))
k <- 2
while (k<length(kp30)) {
k <- k+1
temp1 <- A*(k-2)
temp2 <- ((c^(k-2) - 1)/log(c))*B*c^(30+2)
kp30[k] <- kp30[2]*exp(-temp1-temp2)
}
kp30 <- c(1,kp30)
kp30n <- kp30[-1]
kp30d <- kp30[-length(kp30)]
q30k <- 1 - (kp30n/kp30d)
v <- 1.025/1.05
E3025s <- v^25 * kp30[26]
k <- 0:24
```

```
kp30s <- kp30[1:25]
q30ks <- q30k[1:25]
vk <- v^(k+1)
A30term25s <- sum(vk*kp30s*q30ks)
EZ1 <- (1/1.025)*A30term25s + E3025s

v <- 1/1.05
E3025 <- v^25 * kp30[26]
k <- 0:24
kp30t <- kp30[1:25]
q30kt <- q30k[1:25]
vk <- v^(k+1)
A30term25 <- sum(vk*kp30t*q30kt)
EZ2 <- A30term25 + E3025
```

This produces the results:

```
> A30term25s
[1] 0.01270741
> E3025s
[1] 0.5371817
> EZ1
[1] 0.5495792
> A30term25
[1] 0.008766631
> E3025
[1] 0.2897508
> EZ2
[1] 0.2985174
```

(c) See (b) above for $\mathrm{E}[Z_1]$ and $\mathrm{E}[Z_2]$. Similar calculation for $\mathrm{E}[Z_1^2]$ as $\mathrm{E}[Z_1]$, but at twice the applicable force of interest. In this case, the interest rate effective is

$$i_2 = \left(\frac{1.05}{1.025}\right)^2 - 1.$$

Similarly for $\mathrm{E}[Z_2^2]$, the interest rate effective is

$$i_3 = (1.05)^2 - 1.$$

The R code below shows these calculations:

$$\mathrm{E}[Z_1^2] = 0.3025095 \quad \text{and} \quad \mathrm{E}[Z_2^2] = 0.09019804.$$

```
# repeat process for E[Z1^2] and E[Z2^2]
i2 <- (1.05/1.025)^2 -1
v <- 1/(1+i2)
```

```
E3025s2 <- v^25 * kp30[26]
k <- 0:24
vk <- v^(k+1)
A30term25s2 <- sum(vk*kp30t*q30kt)
EZ1sq <- (1/1.025^2)*A30term25s2 + E3025s2

v <- 1/1.05^2
E3025s3 <- v^25 * kp30[26]
k <- 0:24
vk <- v^(k+1)
A30term25s3 <- sum(vk*kp30t*q30kt)
EZ2sq <- A30term25s3 + E3025s3
```

This produces the results:

```
> EZ1sq
[1] 0.3025095
> EZ2sq
[1] 0.09019804
```

Note that

$$Z_1 Z_2 = \begin{cases} v^{2(K+1)}(1.025)^K, & \text{for } K < 25 \\ v^{2(25)}(1.025)^{25}, & \text{for } K \geq 25 \end{cases}$$

so that its expectation is

$$\mathrm{E}\big[Z_1 Z_2\big] = \frac{1}{1.025} A^{\,1}_{[30]:\overline{25}|i_4} + {}_{25}E_{[30]\,i_4},$$

evaluated at interest rate

$$i_4 = \frac{1.05^2}{1.025} - 1.$$

The R code below shows that

$$\mathrm{E}\big[Z_1 Z_2\big] = \frac{1}{1.025}(0.00629597) + 0.1586313 = 0.1647737$$

with covariance

$$\mathrm{Cov}\big[Z_1, Z_2\big] = \mathrm{E}\big[Z_1 Z_2\big] - \mathrm{E}\big[Z_1\big]\mathrm{E}\big[Z_2\big] = 0.0007147094.$$

```
# repeat process for E[Z1 Z2]
i4 <- (1.05^2/1.025) -1
v <- 1/(1+i4)
E3025s4 <- v^25 * kp30[26]
k <- 0:24
vk <- v^(k+1)
A30term25s4 <- sum(vk*kp30t*q30kt)
EZ1Z2 <- (1/1.025)*A30term25s4 + E3025s4
covZ1Z2 <- EZ1Z2 - EZ1*EZ2
```

This produces the results:

```
> A30term25s4
[1] 0.00629597
> E3025s4
[1] 0.1586313
> EZ1Z2
[1] 0.1647737
> covZ1Z2
[1] 0.0007147094
```

Finally, the variance of the loss-at-issue is therefore

$$\begin{aligned} \mathrm{Var}\left[L_0\right] &= 250000^2 \mathrm{Var}\left[Z_1\right] + \left(\frac{0.99P}{d}\right)^2 \mathrm{Var}\left[Z_2\right] + 2(250000)\left(\frac{0.99P}{d}\right)\mathrm{Cov}\left[Z_1, Z_2\right] \\ &= 250000^2(0.000472206) + \left(\frac{0.99(9764.444)}{1-(1/1.05)}\right)^2 (0.001085407) \\ &\quad +2(250000)\left(\frac{0.99(9764.444)}{1-(1/1.05)}\right)(0.0007147094) \\ &= 146786651. \end{aligned}$$

(d) We calculate the loss-at-issue for various possible values of K. First, we note that at $k = 25$, the loss-at-issue

$$L_0(25) = 250000v^{26}(1.025)^{25} + \frac{0.99P}{d}v^{26} + 1200 + 0.39P - \frac{0.99P}{d} = -10,550.86$$

is a gain to the insurer. Then the insurer is ensured a profit if the insured dies before reaching age 55. To determine the lowest possible remaining lifetime of the insured to ensure a profit, we calculate the loss-at-issue for various values of $k < 25$ as

$$L_0(k) = 250000v^{k+1}(1.025)^k + \frac{0.99P}{d}v^{k+1} + 1200 + 0.39P - \frac{0.99P}{d}.$$

The R code below calculates these losses:

```
# calculate loss-at-issue for various K's
k <- 0:25
temp1 <- 250000*(v^(k+1))*(1.025^k)
temp2 <- (0.99*P/(1-v))*v^(k+1)
temp3 <- 1200 + 0.39*P - (0.99*P/(1-v))
L0k <- temp1 + temp2 + temp3
output <- cbind(k,L0k)
colnames(output) <- c("k","L0(k)")
```

This produces the results:

```
> print(output)
       k      L0(k)
 [1,]  0 233436.571
 [2,]  1 218561.161
 [3,]  2 204259.129
 [4,]  3 190506.385
 [5,]  4 177279.910
 [6,]  5 164557.705
 [7,]  6 152318.748
 [8,]  7 140542.946
 [9,]  8 129211.091
[10,]  9 118304.825
[11,] 10 107806.596
[12,] 11  97699.625
[13,] 12  87967.867
[14,] 13  78595.980
[15,] 14  69569.293
[16,] 15  60873.774
[17,] 16  52496.004
[18,] 17  44423.144
[19,] 18  36642.913
[20,] 19  29143.564
[21,] 20  21913.853
[22,] 21  14943.024
[23,] 22   8220.782
[24,] 23   1737.274
[25,] 24  -4516.931
[26,] 25 -10550.860
```

Thus, we see that the probability the insurer will make a profit is

$$\Pr[K \geq 24] = {}_{24}p_{[30]} = \frac{\ell_{54}}{\ell_{[30]}} = \frac{98022.38}{99721.06} = 0.9829657.$$