Exercise 5.12

(a) The longer you live, the later you die, and vice versa. Hence, the longer you live, the more expensive life annuity would be; the later you die, the cheaper life insurance would be. And of course, vice versa. We therefore expect a negative covariance.

(b) Rewrite the product of Y and Z as

$$YZ = v^T \cdot \bar{a}_{\overline{T|}} = v^T \frac{1 - v^T}{\delta} = \frac{1}{\delta}\left(v^T - v^{2T}\right)$$

so that the covariance can be expressed as

$$\begin{aligned} \mathrm{Cov}[Y, Z] &= \mathrm{E}[YZ] - \mathrm{E}[Y]\mathrm{E}[Z] \\ &= \frac{1}{\delta}\left(\mathrm{E}[v^T] - \mathrm{E}[v^{2T}]\right) - \bar{A}_x \cdot \bar{a}_x \\ &= \frac{1}{\delta}\left(\bar{A}_x - {}^2\bar{A}_x\right) - \bar{A}_x \cdot \bar{a}_x \end{aligned}$$

(c) We can re-express the covariance in (b) as

$$\begin{aligned} \mathrm{Cov}[Y, Z] &= \frac{1}{\delta}\left(\bar{A}_x - {}^2\bar{A}_x\right) - \bar{A}_x\left(\frac{1 - \bar{A}_x}{\delta}\right) \\ &= \frac{1}{\delta}\left(\bar{A}_x - {}^2\bar{A}_x\right) - \frac{1}{\delta}\left[\bar{A}_x - (\bar{A}_x)^2\right] \\ &= -\frac{1}{\delta}\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right] \\ &= -\delta\,\mathrm{Var}\left[\bar{a}_{\overline{T|}}\right]. \end{aligned}$$

This covariance is clearly negative because both δ and $\mathrm{Var}\left[\bar{a}_{\overline{T|}}\right]$ are positive.