Exercise 3.8

(a) Starting with $p_x^* = \ell_{x+1}^*/\ell_x^*$, we note that

$$\ell_x^* = \frac{\ell_{x+1}^*}{p_x^*} = \frac{\ell_{x+1}^*}{p_{x+1}} = \ell_{x+1}^* \frac{\ell_{x+1}}{\ell_{x+2}}.$$

Because $\ell_{25}^* = 98363 = \ell_{26}$, it follows that $\ell_x^* = \ell_{x+1}$. Therefore, we have

$$\ell_{24}^* = \ell_{25} = 98444 \quad \text{and} \quad \ell_{23}^* = \ell_{24} = 98529.$$

Starting with $p_{[x]+2}^* = \ell_{x+3}^*/\ell_{[x]+2}^*$, we note that

$$\ell_{[x]+2}^* = \frac{\ell_{x+3}^*}{p_{[x]+2}^*} = \frac{\ell_{x+3}^*}{{}_2p_{x+2}} = \ell_{x+3}^* \frac{\ell_{x+2}}{\ell_{x+4}},$$

from which it follows that $\ell_{[x]+2}^* = \ell_{x+2}$. Therefore, we have

$$\ell_{[20]+2}^* = \ell_{22} = 98700, \quad \ell_{[21]+2}^* = \ell_{23} = 98615, \quad \text{and} \quad \ell_{[22]+2}^* = \ell_{24} = 98529.$$

Starting with $p_{[x]+1}^* = \ell_{[x]+2}^*/\ell_{[x]+1}^*$, we note that

$$\ell_{[x]+1}^* = \frac{\ell_{[x]+2}^*}{p_{[x]+1}^*} = \frac{\ell_{[x]+2}^*}{{}_3p_{x-1}} = \ell_{[x]+2}^* \frac{\ell_{x-1}}{\ell_{x+2}},$$

from which it follows that $\ell_{[x]+1}^* = \ell_{x-1}$. Therefore, we have

$$\ell_{[20]+1}^* = \ell_{19} = 98942, \quad \ell_{[21]+1}^* = \ell_{20} = 98866, \quad \text{and} \quad \ell_{[22]+1}^* = \ell_{21} = 98785.$$

Finally with $p_{[x]}^* = \ell_{[x]+1}^*/\ell_{[x]}^*$, we note that

$$\ell_{[x]}^* = \frac{\ell_{[x]+1}^*}{p_{[x]}^*} = \frac{\ell_{[x]+1}^*}{{}_4p_{x-5}} = \ell_{[x]+1}^* \frac{\ell_{x-5}}{\ell_{x-1}},$$

from which it follows that $\ell_{[x]}^* = \ell_{x-5}$. Subsequently, we have

$$\ell_{[20]}^* = \ell_{15} = 99180, \quad \ell_{[21]}^* = \ell_{16} = 99135, \quad \text{and} \quad \ell_{[22]}^* = \ell_{17} = 99079.$$

We summarize these results in tabular form below.

x	$\ell_{[x]}^*$	$\ell_{[x]+1}^*$	$\ell_{[x]+2}^*$	ℓ_{x+3}^*	$x+3$
20	99180	98942	98700	98529	23
21	99135	98866	98615	98444	24
22	99079	98785	98529	98363	25

(b)

$$
\begin{aligned}
_{2|38}q^*_{[21]+1} &= {}_2p^*_{[21]+1} - {}_{40}p^*_{[21]+1} = \frac{\ell^*_{24} - \ell^*_{62}}{\ell^*_{[21]+1}} = \frac{98444 - 86455}{98866} = 0.1212651 \\
_{40}p^*_{[22]} &= \frac{\ell^*_{62}}{\ell^*_{[22]}} = \frac{86455}{99079} = 0.8725865 \\
_{40}p^*_{[21]+1} &= \frac{\ell^*_{62}}{\ell^*_{[21]+1}} = \frac{86455}{98866} = 0.8744664 \\
_{40}p^*_{[20]+2} &= \frac{\ell^*_{62}}{\ell^*_{[20]+2}} = \frac{86455}{98700} = 0.8759372 \\
_{40}p^*_{22} &= \frac{\ell^*_{62}}{\ell^*_{22}} = \frac{86455}{98615} = 0.8766922
\end{aligned}
$$