Exercise 2.4

(a) To show S_0 is a legitimate survival function, we show 3 conditions:

(i) $S_0(0) = 1$: trivial

(ii) $\lim_{x\to\infty} S_0(x) = 0$: Since all parameters A, B, C and D are all positive, then the term $Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D} \to \infty$ as $x \to \infty$ so that $\lim_{x\to\infty} S_0(x) = e^{-\infty} = 0$.

(iii) S_0 must be non-increasing: Define the term $H(x) = Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}$ so that $\dfrac{dH(x)}{dx} = A + Bx + CD^x$ and that

$$\frac{dS_0(x)}{dx} = -e^{-H(x)}\frac{dH(x)}{dx},$$

which is clearly strictly negative for all x.

(b) We have

$$\begin{aligned} S_x(t) &= \frac{S_0(x+t)}{S_0(x)} = \frac{\exp\left\{-\left[A(x+t) + B(x+t)^2 + \frac{C}{\log D}D^{x+t} - \frac{C}{\log D}\right]\right\}}{\exp\left\{-\left[Ax + Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right]\right\}} \\ &= \exp\left\{-\left[At + B(2xt + t^2) + \frac{C}{\log D}D^x(D^t - 1)\right]\right\}. \end{aligned}$$

(c) The force of mortality at age x can be expressed as

$$\mu_x = \frac{-dS_0(x)/dx}{S_0(x)} = \frac{e^{-H(x)}\frac{dH(x)}{dx}}{e^{-H(x)}} = A + Bx + CD^x.$$

The force of mortality has a similar form to that of Makeham's except for the addition of a linear term on age x.

(d) Solving all of part (d) requires use of a computer software. Here we give our solution coded in R. [slightly differ from textbook answers]

(i) Note that we can express

$${}_tp_{30} = S_{30}(t) = \exp\left\{-\left[At + B(60t + t^2) + \frac{C}{\log D}D^{30}(D^t - 1)\right]\right\}.$$

The R code to compute this for different values of t is given by

```
A <- 0.00005
B <- 0.0000005
C <- 0.0003
D <- 1.07
tp30 <- function (t) {
temp <- A*t + B*(60*t + t^2) + (C/log(D))*D^30 * (D^t -1)
```

```
exp(-temp)}
t <- c(1,5,10,20,50,90)
output <- cbind(t,round(tp30(t),4))
colnames(output) <- c("t", "tp30")
print(output)
tp30(90)
```

This gives the output

```
> print(output)
      t   tp30
[1,]  1 0.9976
[2,]  5 0.9861
[3,] 10 0.9671
[4,] 20 0.9061
[5,] 50 0.3807
[6,] 90 0.0000
> tp30(90)
[1] 3.497638e-07
```

(ii) Here we note that we can express

$$ {}_tq_{40} = 1 - S_{40}(t) = 1 - \exp\left\{ - \left[At + B(80t + t^2) + \frac{C}{\log D} D^{40}(D^t - 1) \right] \right\}. $$

The R code to compute this for different values of t is given by

```
tq40 <- function (t) {
temp <- A*t + B*(80*t + t^2) + (C/log(D))*D^40*(D^t -1)
1-exp(-temp)}
t <- c(1,10,20)
output <- cbind(t,round(tq40(t),4))
colnames(output) <- c("t", "tq40")
print(output)
```

This gives the output

```
> print(output)
      t   tq40
[1,]  1 0.0047
[2,] 10 0.0631
[3,] 20 0.1751
```

(iii) Here we note that we can express

$$ {}_{t|10}q_{30} = {}_tp_{30} - {}_{t+10}p_{30}. $$

This gives the probability that a life (30) will survive the next t years but dies the following 10 years after that. The R code to compute this for different values of t is given by

```
tbar10q30 <- function (t) {
temp1 <- tp30(t)
temp2 <- tp30(t+10)
temp1-temp2}
t <- c(1,10,20)
round(tbar10q30(t),4)
output <- cbind(t,round(tbar10q30(t),4))
colnames(output) <- c("t", "t|10q30")
print(output)
```

This gives the output

```
> print(output)
      t t|10q30
[1,]  1  0.0351
[2,] 10  0.0610
[3,] 20  0.1084
```

(iv) We evaluate the curtate expectation of life at age x using $e_x = \sum_{k=1}^{\infty} {}_kp_x$. The logic in the R code is to keep summing the term ${}_kp_x$ until a certain level of very small tolerance. Here we choose our tolerance to be 10^{-50}, indeed a very small value. The R code to compute this for different values of x is given by

```
kpx <- function (k,x) {
temp <- A*k + B*(2*x*k + k^2) + (C/log(D))*D^x * (D^k -1)
exp(-temp)}
ex <- function(x,tol) {
k<-1
p1 <- kpx(k,x)
p <- p1
while (p1 > tol) {
k <- k+1
p1 <- kpx(k,x)
p <- p + p1}
p}
x <- 70:75
expd <- rep(0,6)
tol <- 10^(-50)
expd[1] <- ex(x[1],tol)
expd[2] <- ex(x[2],tol)
expd[3] <- ex(x[3],tol)
expd[4] <- ex(x[4],tol)
expd[5] <- ex(x[5],tol)
expd[6] <- ex(x[6],tol)
output <- cbind(x,round(expd,3))
colnames(output) <- c("x", "ex")
print(output)
```

This gives the output

```
> print(output)
      x     ex
[1,] 70 13.041
[2,] 71 12.513
[3,] 72 11.997
[4,] 73 11.495
[5,] 74 11.005
[6,] 75 10.529
```

(v) Finally, to evaluate the complete expectation of life at age x, we numerically approximate the integral $\mathring{e}_x = \int_0^\infty {}_tp_x dt$. To approximate this integral, we use repeated application of Simpson's rule given in Appendix B of the book:

$$\int_a^{a+2h} {}_tp_x dt \approx \frac{h}{3}[{}_ap_x + 4{}_{a+h}p_x + {}_{a+2h}p_x],$$

starting with $a = 0$ and choosing $h = 0.25$. We repeat and calculate additively the integral for consecutive intervals of length $2h$, until a certain level of very small tolerance. Here we choose $h = 0.25$ and our tolerance to be 10^{-50}. The R code to compute this for different values of x is given by

```
tpx <- function (t,x) {
temp <- A*t + B*(2*x*t + t^2) + (C/log(D))*D^x * (D^t -1)
exp(-temp)}
exc <- function(x,tol) {
a<-0
h<-.25
k<-0
v1 <- (h/3)*(tpx(a,x) + 4* tpx(a+h,x) + tpx(a+2*h,x))
v <- v1
while (v1 > tol) {
k <- k+2
lim1 <- a+k*h
mid <- a+(k+1)*h
lim2 <- a+(k+2)*h
v1 <- (h/3)*(tpx(lim1,x) + 4* tpx(mid,x) + tpx(lim2,x))
v <- v + v1}
v}
x <- 70:75
expc <- rep(0,6)
tol <- 10^(-50)
expc[1] <- exc(x[1],tol)
expc[2] <- exc(x[2],tol)
expc[3] <- exc(x[3],tol)
expc[4] <- exc(x[4],tol)
expc[5] <- exc(x[5],tol)
```

```
expc[6] <- exc(x[6],tol)
output <- cbind(x,round(expc,3))
colnames(output) <- c("x", "exc")
print(output)
```

This gives the output

```
> print(output)
      x    exc
[1,] 70 13.539
[2,] 71 13.010
[3,] 72 12.494
[4,] 73 11.991
[5,] 74 11.501
[6,] 75 11.025
```