Exercise 2.2

(a) The implied limiting age ω is the solution to $G(\omega) = 0$ which leads us to

$$18000 - 110\omega - \omega^2 = -(\omega - 90)(\omega + 200) = 0.$$

Thus, $\omega = 90$ since the limiting age cannot be negative.

(b) For G to be a legitimate survival function, it must satisfy 3 conditions:

(i) $G(0) = 1$: trivial

(ii) $G(\omega) = 0$: verified in (a) above.

(iii) G must be non-increasing. We check whether $dG(x)/dx \leq 0$.

$$\frac{dG(x)}{dx} = \frac{-2(55 + x)}{18000}$$

which clearly is non-positive for all $0 \leq x \leq 90$.

(c) Now that we have verified $G(x)$ is a legitimate survival function, we can write it as $S_0(x)$ so that

$${}_{20}p_0 = \Pr[T_0 > 20] = S_0(20) = \frac{18000 - 110(20) - 20^2}{18000} = \frac{15400}{18000} = \frac{77}{90} = 0.8555556.$$

This gives the probability that a newborn will survive to age 20.

(d) The survival function for a life age 20 can be expressed as

$$\begin{aligned} S_{20}(t) &= \Pr[T_{20} > t] = \frac{\Pr[T_0 > 20 + t]}{\Pr[T_0 > t]} = \frac{S_0(20 + t)}{S_0(20)} \\ &= \frac{[18000 - 110(20 + t) - (20 + t)^2]/18000}{[18000 - 110(20) - 20^2]/18000} \\ &= \frac{[18000 - 110(20) - 110t - 20^2 - 40t - t^2]/18000}{[18000 - 110(20) - 20^2]/18000} = 1 - \frac{150t + t^2}{15400}. \end{aligned}$$

(e) The probability that (20) will die between the ages of 30 and 40 is

$$\begin{aligned} \Pr[10 < T_{20} < 20] &= S_{20}(10) - S_{20}(20) = \frac{150(20) + 20^2}{15400} - \frac{150(10) + 10^2}{15400} \\ &= \frac{1800}{15400} = \frac{9}{77} = 0.1168831. \end{aligned}$$

(f) The force of mortality at age x is given by

$$\mu_x = \frac{-dS_0(x)/dx}{S_0(x)} = \frac{[110 + 2x]/18000}{[18000 - 110x - x^2]/18000} = \frac{110 + 2x}{18000 - 110x - x^2},$$

so that $\mu_{50} = \dfrac{110 + 2(50)}{18000 - 110(50) - 50^2} = \dfrac{21}{1000} = 0.021.$