## Exercise 2.15

(a) We know that

$$\mathring{e}_x = \int_0^\infty {}_sp_x ds = \int_0^\infty \frac{S_0(x+s)}{S_0(x)} ds = \frac{1}{S_0(x)} \int_0^\infty S_0(x+s) ds.$$

Using a change of variable of integration $t = x + s$, we find that

$$\mathring{e}_x = \frac{1}{S_0(x)} \int_0^\infty S_0(x+t) dt = \frac{1}{S_0(x)} \int_x^\infty S_0(t) dt$$

and the result follows. Now taking the derivative of both sides with respect to $x$, we find

$$\frac{d}{dx}\mathring{e}_x = \frac{-S_0(x)S_0(x) + f_0(x)\displaystyle\int_x^\infty S_0(t)dt}{S_0(x)^2} = -1 + \frac{f_0(x)}{S_0(x)} \cdot \frac{\displaystyle\int_x^\infty S_0(t)dt}{S_0(x)}.$$

The result follows because we know that

$$\mu_x = \frac{f_0(x)}{S_0(x)}$$

and

$$\mathring{e}_x = \frac{1}{S_0(x)} \int_x^\infty S_0(t) dt.$$

Another approach to prove this is to use the result of Exercise 2.9:

$$\frac{d}{dx}\mathring{e}_x = \int_0^\infty {}_tp_x(\mu_x - \mu_{x+t} dt = \mu_x \int_0^\infty {}_tp_x dt - \int_0^\infty {}_tp_x \mu_{x+t} dt = \mu_x \mathring{e}_x - 1.$$

(b) If we let $g(x) = x + \mathring{e}_x$, then

$$\frac{d}{dx} g(x) = 1 + \frac{d}{dx}\mathring{e}_x = 1 + \mu_x \mathring{e}_x - 1 = \mu_x \mathring{e}_x > 0.$$

Thus, $g$ is an increasing function of age $x$. This means that as you age, the higher your average age at death. Each year you survive is an addition to your average age at death for certain.