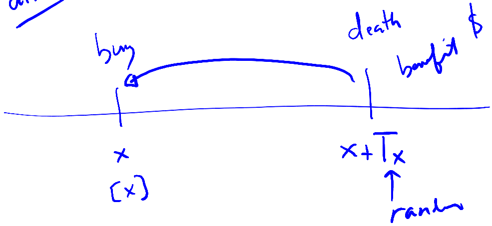
Insurance Benefits

Lecture: Weeks 6-8

PV discounted



An introduction

- Central theme: to quantify the value today of a (random) amount to be paid at a random time in the future.
  - main application is in life insurance contracts, but could be applied in other contexts, e.g. warranty contracts.
- Generally computed in two steps:

  benefit discount factor

  1. take the present value (PV) random variable, $b_T v_T$; and
  2. calculate the expected value $\mathrm{E}[b_T v_T]$ for the average value - this value is referred to as the Actuarial Present Value (APV).
- In general, we want to understand the entire distribution of the PV random variable $b_T v_T$:
  - it could be highly skewed, in which case, there is danger to use expectation.
  - other ways of summarizing the distribution such as variances and percentiles/quantiles may be useful.

traditional -

A simple illustration

Consider the simple illustration of valuing a three-year term insurance policy issued to age 35 where if he dies within the first year, a \$1,000 benefit is payable at the end of his year of death.

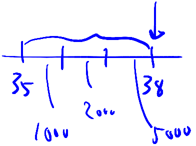
If he dies within the second year, a \$2,000 benefit is payable at the end of his year of death. If he dies within the third year, a \$5,000 benefit is payable at the end of his year of death.

Assume a constant discount rate of 5% and the following extract from a mortality table:

d

$i = 5\%$

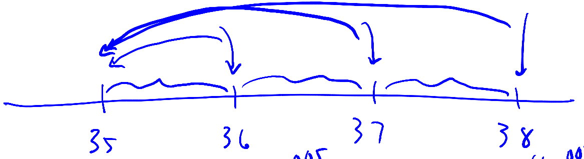
x	q_x
35	0.005
36	0.006
37	0.007
38	0.008



b

Calculate the APV of the benefits.

$v = \frac{1}{1+i}$,



$$\text{APV(benefits)} = 1000\, \underset{\frac{1}{1.05}}{v} \cdot \overset{.005}{q_{35}} + 2000\, v^2\, \overset{(1-.005)}{p_{35}}\, \overset{.006}{q_{36}}$$

$$+\, 5000\, v^3\, \underset{(1-.005)}{p_{35}}\, \underset{(1-.006)}{p_{36}}\, \underset{.007}{q_{37}}$$

$$= \underline{\underline{45.49448}}$$

Chapter summary

- Life insurance
  - benefits payable contingent upon death; payment made to a designated beneficiary
  - actuarial present values (APV)
  - actuarial symbols and notation
- Insurances payable at the moment of death
  - continuous
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Insurances payable at the end of year of death
  - discrete
  - level benefits, varying benefits (e.g. increasing, decreasing)
- Chapter 4 (Dickson, et al.) - both 1st/2nd ed.

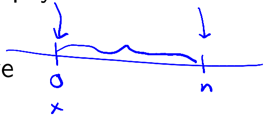
The present value random variable

- Denote by Z, the present value random variable.
- This gives the value, at policy issue, of the benefit payment. Issue age is usually denoted by x. $_{}$ or $[x]$
- In the case where the benefit is payable at the moment of death, Z clearly depends on the time-until-death T. For simplicity, we drop the subscript x for age-at-issue.
- It is $Z = b_T v_T$ where:
  - b_T is called the benefit payment function
  - v_T is the discount function
- In the case where we have a constant (fixed) interest rate, then $v_T = v^T = (1+i)^{-T} = e^{-\delta T}$. $v = e^{-\delta} = \frac{1}{1+i} = 1-d$

Fixed term life insurance

n-year term LI

- An n-year term life insurance provides payment if the insured dies within n years from issue.

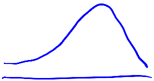


- For a unit of benefit payment, we have

$$b_T = \begin{cases} 1, & T \leq n \\ 0, & T > n \end{cases} \text{ and } v_T = v^T.$$

$$I(\cdot) = \begin{cases} 1 & \text{if true} \\ 0 & \text{if false} \end{cases}$$

- The present value random variable is therefore



$$Z = \begin{cases} v^T, & T \leq n \\ 0, & T > n \end{cases} = v^T I(T \leq n)$$

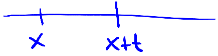
where $I(\cdot)$ is called indicator function. E$[Z]$ is called the APV of the insurance.

- Actuarial notation:

$$\bar{A}^{\ 1}_{x:\overline{n}|} = \mathsf{E}[Z] = \int_0^n v^t f_x(t)dt = \int_0^n v^t {}_tp_x \mu_{x+t} dt.$$

r.v. $I(T \leq n) \Rightarrow$ binary $\quad E(I(T\leq n)) = Pr(T \leq n) = {}_nq_x$

$$E[v^T I(T\leq n)] = \int_0^\infty v^t \underbrace{I(t\leq n)}\, \underbrace{f_x(t)}_{{}_tp_x \mu_{x+t}}\, dt$$



$$\bar{A}^{1}_{x:\overline{n}|} = \int_0^n v^t\, {}_tp_x\, \mu_{x+t}\, dt$$

Benefit = \$1

Constant δ

constant μ

$v^t \to e^{-\delta t}$, $\quad {}_tp_x \to e^{-\mu t}$, $\quad \mu_{x+t} \to \mu$

$\underbrace{e^{-\mu t} \quad \mu}_{\text{Exponential}}$

$$\bar{A}^{1}_{x:\overline{n}|} = \mu \int_0^n e^{-(\delta+\mu)t}\, dt = \frac{\mu}{\mu+\delta}\left(-e^{-(\delta+\mu)t}\Big|_0^n \right)$$

$$= \frac{\mu}{\mu+\delta}\left(1 - e^{-(\delta+\mu)n}\right) \to \qquad n \to \infty$$

Rule of moments

- The j-th moment of the distribution of Z can be expressed as:

$Z = v^{T} I(T \leq n)$

$Z^{j} = v^{jT} I(T \leq n)$

$$\mathsf{E}\left[Z^j\right] = \int_0^n v^{tj}\,{}_tp_x \mu_{x+t} dt = \underbrace{\int_0^n e^{-(j\delta)t}\,{}_tp_x \mu_{x+t} dt.}_{{}^{j}\bar{A}^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} @ j\delta}$$

- This is actually equal to the APV but evaluated at the force of interest $j\delta$.
- In general, we have the following rule of moment:

$$\mathsf{E}\left[Z^j\right]_{@\ \delta_t} = \mathsf{E}[Z]_{@\ j\delta_t}.$$

- For example, the variance can be expressed as term insurance

$$\mathsf{E}[Z^2] - (\mathsf{E}[Z])^2 = \mathrm{Var}[Z] = {}^{2}\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2.$$

2δ

Term Insurance:

constant δ, constant μ } assumptions

$$E[Z] = \frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right)$$

$$E[Z^2] = \text{rule of moment @ } 2\delta = \underbrace{\frac{\mu}{\mu+2\delta}\left(1 - e^{-(\mu+2\delta)n}\right)}_{{}^2\bar{A}^{1}_{x:\overline{n}|}}$$

Practice: $\delta = .05$ 10-year term benefit $= 1$

$\mu = .001$

$$APV(\text{benefits}) = \frac{.001}{.051}\left(1 - e^{-.051(10)}\right) = \underline{.00783742}$$

$$Var(Z) = E[Z^2] - (.00783742)^2 = .00629486\underline{2} - (.00783742)^2$$

$$= \underline{.00623349\underline{9}}$$

$$2\delta \quad \frac{.001}{.101}\left(1 - e^{-.101(10)}\right) = {}^2\bar{A}^{1}_{x:\overline{n}|} = .00629486\underline{2}$$

$B =$ benefit, say

$$APV(\text{benefit}) = B \cdot \bar{A}^{1}_{x:\overline{n}|}$$

$$Var(Z) = Var(B \cdot {}^{\$1}\text{P.V.}) = B^2 \left[{}^2\bar{A}^{1}_{x:\overline{n}|} - \left(\bar{A}^{1}_{x:\overline{n}|}\right)^2 \right]$$

Whole life insurance

$n \to \infty$

- For a whole life insurance, benefits are payable following death at any time in the future.
- Here, we have $b_T = 1$ so that the present value random variable is $Z = v^T$.
- $= \cancel{\bar{A}_{x:\overline{\infty}|}}$

  APV notation for whole life: $\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t \, {}_tp_x \mu_{x+t} dt$.
- Variance (using rule of moments):

$$\mathrm{Var}[Z] = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2.$$

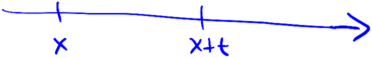
- Whole life insurance is the limiting case of term life insurance as $n \to \infty$.
- Note also that if the benefit amount is not 1, but say $b_T = b$, then $\mathrm{E}[Z] = b\bar{A}_x$ and that $\mathrm{Var}[Z] = b^2 \left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$.

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, dt$$

Constant δ

constant μ

$$v^t \to e^{-\delta t}, \quad {}_tp_x \to e^{-\mu t}, \quad \mu_{x+t} \to \mu$$

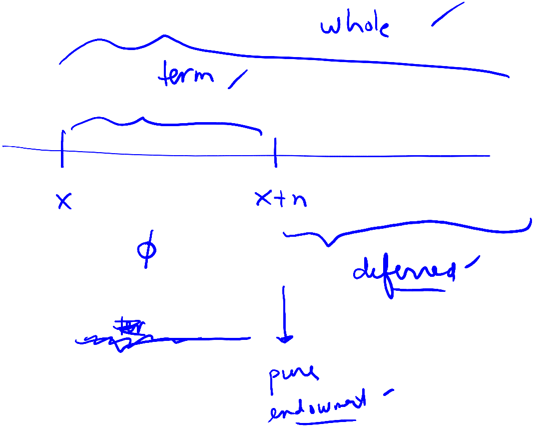


$$= \mu \int_0^\infty e^{-(\delta+\mu)t} \, dt = \frac{\mu}{\mu+\delta}\left(-e^{-(\mu+\delta)t}\Big|_0^\infty\right)$$

$$= \frac{\mu}{\mu+\delta},$$

$${}^2\bar{A}_x = \frac{\mu}{\mu+2\delta},$$

$$\mathrm{Var}[Z] = \frac{\mu}{\mu+2\delta} - \left(\frac{\mu}{\mu+\delta}\right)^2 \geq 0$$


whole
term
x
x+n
ϕ
deferred
pure
endowment

Pure endowment insurance

- For an n-year pure endowment insurance, a benefit is payable at the end of n years if the insured survives at least n years from issue.
- Here, we have $b_T = \begin{cases} 0, & T \le n \\ 1, & T > n \end{cases}$ and $v_T = v^n$ so that the PV r.v. is

$$Z = \begin{cases} 0, & T \le n \\ v^n, & T > n \end{cases}. \quad = v^n I(T>n)$$

$E(Z)$ $\quad Var(v^n I(T>n))$

$v^{2n} Var(I(T>n))$

- APV for pure endowment: $A_{x:\overset{\,1}{\overline{n|}}} = {}_nE_x = v^n\, {}_np_x$.
- Variance (using rule of moments):

Rule of moment

$$\mathrm{Var}[Z] = v^{2n}\, {}_np_x \cdot {}_nq_x = {}^2A_{x:\overset{\,1}{\overline{n|}}} - \left(A_{x:\overset{\,1}{\overline{n|}}}\right)^2.$$

- Sometimes, we can also express the present value random variable based on an indicator function:

$$Z = v^n I(T_x > n),$$

where $I(E)$ is 1 if the event E is true, and 0 otherwise.

Pure Endowment

$$Z = v^n I(T>n)$$

$$E[Z] = E[v^n I(T>n)] = v^n \underbrace{\underbrace{Pr(T>n)}_{{}_n p_x}}_{\text{APV of an n-year pure endowment}}$$

\$1 benefit

$$= A_{x:\overset{1}{\overline{n|}}}$$

$$Var[Z] = Var[v^n I(T>n)]$$

$$= v^{2n} \cdot {}_n p_x (1 - {}_n p_x)$$

Since $I(T>n) = \begin{cases} 1 \text{ or} \\ 0 \end{cases}$ binary.

constant δ, constant μ >

$$A_{x:\overset{1}{\overline{n|}}} = v^n {}_n p_x = e^{-\delta n} e^{-\mu n} = e^{-(\mu+\delta)n}$$

$$Var[Z] = v^{2n} {}_n p_x (1 - {}_n p_x) = e^{-2\delta n} e^{-\mu n} (1 - e^{-\mu n})$$

$$ {}^2A_{x:\overset{1}{\overline{n|}}} - \underbrace{\left(A_{x:\overset{1}{\overline{n|}}} \right)^2} $$

$$ \underline{e^{-(\mu+2\delta)n}} - \left(e^{-(\mu+\delta)n} \right)^2 $$

$$ e^{-2\delta n} e^{-\mu n} \left(1 - e^{-\mu n} \right) $$

Rule of Moment also applies for pure endowment !

Endowment insurance

- For an n-year endowment insurance, a benefit is payable if death is within n years or if the insured survives at least n years from issue, whichever occurs first.
- Here, we have $b_T = 1$ and $v_T = \begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases}$ so that the PV r.v. is

$$Z = \begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases}.$$

- It is easy to see that we can re-write Z as $Z = v^{\min(T,n)}$.
- APV endowment: $\bar{A}_{x:\overline{n|}} = \bar{A}^{\,1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\,1}$.

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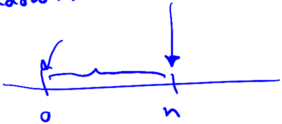
- Variance (using rule of moments):

$$\text{Var}[Z] = {}^2\bar{A}_{x:\overline{n|}} - \left(\bar{A}_{x:\overline{n|}}\right)^2.$$

1. term insurance
2. whole life insurance $\quad n \to \infty$
3. pure endowment
4. endowment = term + endowment

$$b_T = \begin{cases} 1, & T \leq n \\ 1, & T > n \end{cases}$$



$$v_T = \begin{cases} v^T, & T \leq n \\ v^n, & T > n \end{cases}$$

$$Z = \underbrace{\begin{cases} v^T, & T \leq n \\ v^n, & T > n \end{cases}}_{= v^T I(T \leq n) + v^n I(T > n)} = \underbrace{\begin{cases} v^T, & T \leq n \\ 0, & T > n \end{cases}}_{Z_1} + \underbrace{\begin{cases} 0, & T \leq n \\ v^n, & T > n \end{cases}}_{Z_2}$$

$$Z^2 = (\qquad\qquad)^2$$

$$Z = Z_1 + Z_2$$

$$Var(Z) = Var(Z_1) + Var(Z_2) + 2\,Cov(Z_1, Z_2)$$

rule of moment   rule of moments

$$Cov(Z_1, Z_2) = \cancel{E(Z_1 Z_2)}^{0} - E(Z_1)\,E(Z_2) = -\bar{A}^{\,1}_{x:\overline{n}|}\,A_{x:\overline{n}|}^{\;\;1}$$

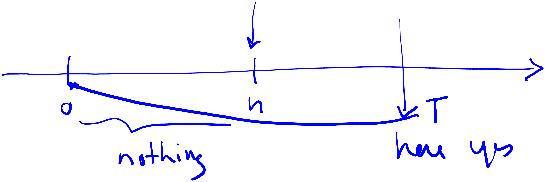
Constant δ
constant μ

$$E(Z) = E(Z_1) + E(Z_2) = \bar{A}^{\,1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1} = \bar{A}_{x:\overline{n}|}$$

$$\frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right) \qquad e^{-(\mu+\delta)n}$$

Deferred insurance

- For an n-year deferred whole insurance, a benefit is payable if the insured dies at least n years following issue.
- Here, we have $b_T = \begin{cases} 0, & T \leq n \\ 1, & T > n \end{cases}$ and $v_T = v^T$ so that the PV r.v. is
$$Z = \begin{cases} 0, & T \leq n \\ v^T, & T > n \end{cases}.$$
- APV for deferred insurance: ${}_{n|}\bar{A}_x = \int_n^\infty v^t \, {}_t p_x \mu_{x+t} dt.$
- Variance (using rule of moments):
$$\mathrm{Var}[Z] = {}_{n|}^{2}\bar{A}_x - \left({}_{n|}\bar{A}_x\right)^2.$$



n-year deferred insurance

$$b_T = \begin{cases} 0, & T \le n \\ 1, & T > n \end{cases} \qquad v_T = \begin{cases} 0, & T \le n \\ v^T, & T > n \end{cases}$$

$$Z = b_T v_T = \begin{cases} 0, & T \le n \\ v^T, & T > n \end{cases} = v^T I(T > n)$$

$$E(Z) = APV = {}_{n|}\bar{A}_x$$

$$Z^2 = \underbrace{v^{2T} I(T > n)}_{2\delta \text{ for rule of moments}}$$

$$Var(Z) = {}^2_{n|}\bar{A}_x - ({}_{n|}\bar{A}_x)^2$$

Constant δ

Constant μ

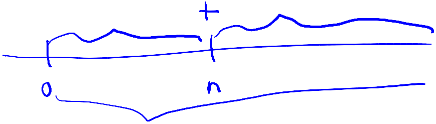
$$Z = \begin{cases} 0, & T \leq n \\ v^T, & T > n \end{cases}$$

$$E(Z) = \int_n^\infty \underbrace{v^t}_{e^{-\delta t}} \, \overbrace{{}_tp_x}^{e^{-\mu t}} \, \overbrace{\mu_{x+t}}^{\mu} \, dt = \frac{\mu}{\mu+\delta}\left(-e^{-(\mu+\delta)t} \Big|_n^\infty \right)$$

$$= \frac{\mu}{\mu+\delta} e^{-(\mu+\delta)n}$$

$$E(Z^2) = \frac{\mu}{\mu+2\delta} e^{-(\mu+2\delta)n}$$

n-year term + n-year deferred = whole life

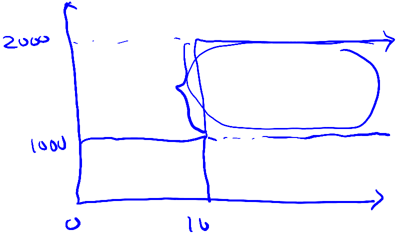


Varying benefits to (x)

Look benefits of say

1000 if you die in the first 10 years

2000 if you die after 10 years



APV of this insurance

$$= 2000\,\bar{A}_x - 1000\,\bar{A}^{1}_{x:\overline{10|}}$$

$$= 1000\,\bar{A}_x + 1000\,{}_{n|}\bar{A}_x$$

Constant force of mortality - all throughout life

Assume mortality is based on a constant force, say μ, and interest is also based on a constant force of interest, say δ.

- Find expressions for the APV for the following types of insurances:
  - whole life insurance; $\bar{A}_x = \mu / \mu+\delta$
  - *n*-year term life insurance; $\bar{A}^{1}_{x:\overline{n}|} = \frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right)$
  - *n*-year endowment insurance; and $\bar{A}_{x:\overline{n}|} = \frac{\mu}{\mu+\delta}\left(1 - e^{-(\mu+\delta)n}\right) + e^{-(\mu+\delta)n}$
  - *m*-year deferred life insurance. ${}_{m|}\bar{A}_x = \frac{\mu}{\mu+\delta} e^{-(\mu+\delta)m}$
- Check out the (corresponding) variances for each of these types of insurance.

[Details in class]

De Moivre's law

Find expressions for the APV for the same types of insurances in the case where you have:

- De Moivre's law.

$T_x \sim \text{Uniform}(0, \omega - x)$

$\omega =$ limiting age

$f_x(t) = \frac{1}{\omega - x}, \quad 0 \leq t \leq \omega - x$

whole life: $\bar{A}_x = \int_0^{\cancel{\infty}\ \omega-x} v^t \underbrace{f_x(t)}_{\frac{1}{\omega-x}} dt = \frac{1}{\omega-x} \underbrace{\int_0^{\omega-x} v^t\, dt}_{\bar{a}_{\overline{\omega-x}|} = \frac{1-v^{\omega-x}}{\delta}}$

term: $\bar{A}^{1}_{x:\overline{n}|} = \int_0^n v^t \frac{1}{\omega-x} dt = \frac{1}{\omega-x} \bar{a}_{\overline{n}|}, \quad n \leq \omega-x$

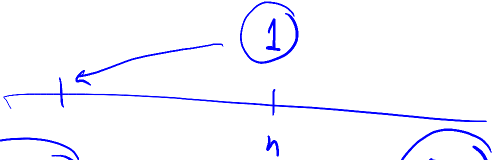
pure endowmt: $A_{x:\overline{n}|}^{\ \ 1} = \underbrace{v^n}_{e^{-\delta n}}\ {}_np_x =$

$\Pr[T_x > n] = \int_n^{\omega-x} \frac{1}{\omega-x} dz = 1 - \frac{n}{\omega-x}$,

endowmt: $\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$

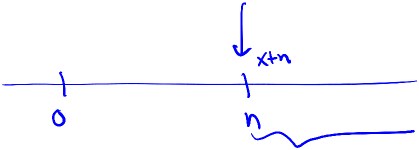
deferred: ${}_{n|}\bar{A}_x = \int_n^{\omega-x} v^t \frac{1}{\omega-x} dt = \frac{1}{\omega-x}\left(\bar{a}_{\overline{\omega-x}|} - \bar{a}_{\overline{n}|} \right)$

$\overset{?}{=} {}_nE_x\, \bar{A}_{x+n}$ (exercise!)



$v^n \, {}_np_x$ = pure endowment = ${}_nE_x$

$= A_{x:\overline{n}|}^{\;\;1}$



$${}_{n|}\bar{A}_x = \underbrace{{}_nE_x \, \bar{A}_{x+n}}$$

Illustrative example 1

For a whole life insurance of \$1,000 on (x) with benefits payable at the moment of death, you are given:

$$\delta_t = \begin{cases} 0.04, & 0 < t \leq 10 \\ 0.05, & t > 10 \end{cases}$$

and

$$\mu_{x+t} = \begin{cases} 0.006, & 0 < t \leq 10 \\ 0.007, & t > 10 \end{cases}$$

Calculate the actuarial present value for this insurance.

$1000\,\bar{A}_x$ — δ is constant

1000

$0 \qquad\qquad 10$

$\delta: \quad .04 \qquad\qquad .05$

$\mu: \quad .006 \qquad\qquad .007$

APV(benefits)

= APV(insurance)

$$= \left[\frac{.006}{.046}\left(1 - e^{-.046(10)}\right) + \underbrace{v^n\,{}_np_x}_{\text{pure endowment}} \frac{.007}{.057} \right] * 1000$$

$$e^{-.04(10)}\, e^{-.006(10)} \left(\frac{.007}{.057}\right)$$

$= 125.6195$

$E(v^T) \neq v^{E(T)}$

Example $\delta = .03$

$$\mu_{x+t} = \begin{cases} .001, & 0 \leq t \leq 5 \\ .002, & t > 5 \end{cases}$$

$$b_t = \begin{cases} 10, & 0 \leq t \leq 5 \\ 20, & t > 5 \end{cases}$$

APV (benefits)

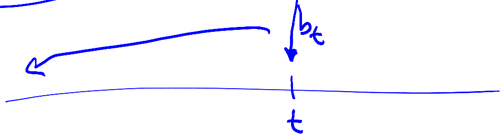
$$= 10 * \frac{.001}{.031}\left(1 - e^{-.031(5)}\right)$$

$$\underbrace{+ 20 * e^{-.031(5)} \frac{.002}{.032}}$$

$$= \underline{??}$$

		0	5	
δ:		.03		.03
μ		.001		.002
b_t		10		20

$$APV = \boxed{\int_0^\infty b_t v^t \, {}_tp_x \, \mu_{x+t} \, dt}$$



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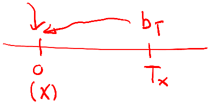
THIS IS WHERE
TOPICS FOR
CLASS TEST 1
ENDS

\$100

Equivalent probability calculations



We can also compute probabilities of Z as follows. Consider the present value random variable Z for a whole life issued to age x. For $0 < \alpha < 1$, the following is straightforward:

$$\begin{aligned} \Pr[Z \leq \alpha] &= \Pr[e^{-\delta T_x} \leq \alpha = \Pr[-\delta T_x \leq \log(\alpha)] \\ &= \Pr[T_x > -(1/\delta)\log(\alpha)] = {}_u p_x, \end{aligned}$$

where

$$u = (1/\delta)\log(1/\alpha) = \log(1/\alpha)^{1/\delta}.$$

- Consider the case where $\alpha = 0.75$ and $\delta = 0.05$. Then $u = \log(1/0.75)^{1/0.05} = 5.753641$.
- Thus, the probability $\Pr[Z \leq 0.75]$ is equivalent to the probability that (x) will survive for another 5.753641 years.

\$100

10

$Z = PV \le \alpha = \text{known}$

$b_T = 1$ $\quad Z = v^{T_x} = e^{-\delta T_x}$

$\underbrace{\Pr[Z \le \alpha]}$ $\iff \Pr[T_x \ge \cancel{d}\text{time}]$

$e^{-\delta T_x} \le \alpha \iff -\delta T_x \le \log(\alpha)$

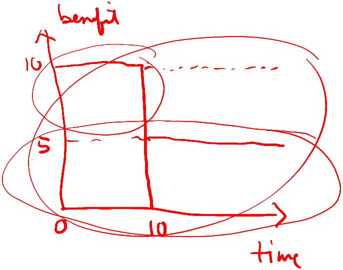
$\iff T_x \ge -\frac{1}{\delta}\log(\alpha) = u$

$\Pr[Z \le \alpha] \iff \underbrace{\Pr[T_x \ge u]} = {}_u p_x$ = Prob survive u more years

$Z > \alpha \iff T_x \le u = {}_u q_x$

$$\text{Benefit} = \begin{cases} 10 & \text{if death in 1st 10 years} \\ 5 & \text{if death after 10 years} \end{cases}$$

whole life to (x)



APV of this

$$10\, \bar{A}^{\,1}_{x:\overline{10}|} + 5\, {}_{10|}\bar{A}_x$$

$$= 5\, \bar{A}_x + 5\, \bar{A}^{\,1}_{x:\overline{10}|}$$

$$= 10\, \bar{A}_x - 5\, {}_{10|}\bar{A}_x$$

Insurances with varying benefits

Type	b_T	Z	APV
Increasing whole life	$\lfloor T+1 \rfloor$	$\lfloor T+1 \rfloor v^T$	$(I\bar{A})_x$
Whole life increasing m-thly	$\lfloor Tm+1 \rfloor /m$	$v^T \lfloor Tm+1 \rfloor /m$	$(I^{(m)}\bar{A})_x$
Constant increasing whole life	T	Tv^T	$(\bar{I}\bar{A})_x$
Decreasing n-year term	$\begin{cases} n - \lfloor T \rfloor, & T \leq n \\ 0, & T > n \end{cases}$	$\begin{cases} (n - \lfloor T \rfloor)\, v^T, & T \leq n \\ 0, & T > n \end{cases}$	$(D\bar{A})^{1}_{x:\overline{n\rvert}}$

* These items will be discussed in class.

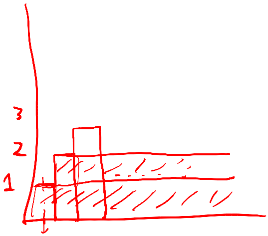
Increasing Whole Life

increases each year by a fixed amount

$Z = b_T v^T$

$b_T \to \lfloor T+1 \rfloor$

1, 2,



$E[Z] = (I\bar{A})_x = \underbrace{\int_0^\infty \lfloor t+1 \rfloor v^t \, {}_tp_x \mu_{x+t} \, dt}$ $\quad \lfloor x \rfloor$ = greatest integer $\leq x$

$(I\bar{A})_x \to$ increasing

$= \sum_{k=1}^{\infty} k \bar{A}^{1}_{x:\overline{k|}}$ $\quad$ sum of term

$= \bar{A}_x + {}_{1|}\bar{A}_x + {}_{2|}\bar{A}_x + \cdots$

sum of deferred

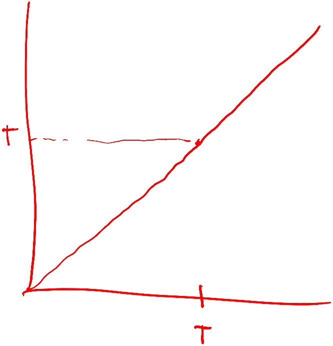
$$b_T = T$$

$$v_T = v^T$$

$$\underbrace{\int_0^\infty t\, v^t \, {}_tp_x \, \mu_{x+t} \, dt}$$

$$= E\left(Z = T v^T\right) =$$ constant increasing whole life

$$= (\bar{I}\bar{A})_x$$



$$b_T = \left\lfloor \frac{mT+1}{m} \right\rfloor = \begin{cases} 1/m \\ 2/m \\ 3/m \\ \vdots \end{cases}$$

$$v_T = v^T$$

$$\underbrace{E\left(Z = \left\lfloor \frac{mT+1}{m} \right\rfloor v^T\right)} = \text{APV}$$

$$(I^{(m)}\bar{A})_x$$

frequency of payment

$m = 12$ monthly

$= 2$ semi-annual

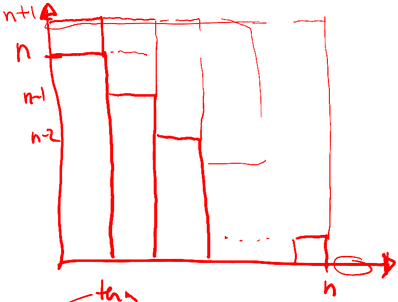
$= 3$ quarterly

$= 4$ $3\times/\text{yr}$

$$m \to \infty \quad (\bar{I}\bar{A})_x$$

Decreasing term insurance

$$b_T = \begin{cases} n - \lfloor T \rfloor, & T \leq n \\ \phi, & T > n \end{cases}$$



v^T

$$E(b_T v^T = Z) = (D\bar{A})^{1}_{x:\overline{n}|}$$

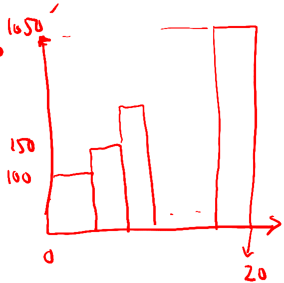
term

policy period

$$\boxed{(I\bar{A})^{1}_{x:\overline{n}|} + (D\bar{A})^{1}_{x:\overline{n}|} = (n+1)\bar{A}^{1}_{x:\overline{n}|}}$$

Benefit of say 100 1st year increasing by 50 thereafter to (x) for 20 years

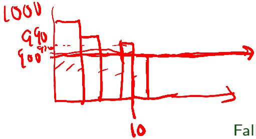
term insurance of 20 years



APV in symbols $= 50\,\bar{A}^{1}_{x:\overline{20}|} + 50\,(I\bar{A})^{1}_{x:\overline{20}|}$

$$\neq 100\,(I\bar{A})^{1}_{x:\overline{20}|}$$

$$\text{APV} = 900\,\bar{A}_x + 10\,(D\bar{A})^{1}_{x:\overline{10}|}$$



Illustrative example 2

50 110

60 years left

For a whole life insurance on (50) with death benefits payable at the moment of death, you are given:

$f(t) = \frac{1}{60}, \quad t \leq 60$

- Mortality follows De Moivre's law with $\omega = 110$.
- $b_t = 10000(1.10)^t$, for $t \geq 0$
- $\delta = 5\%$
- Z denotes the present value random variable for this insurance.

Calculate E$[Z]$ and Var$[Z]$.

Can you find an explicit expression for the distribution function of Z, i.e. Pr$[Z \leq z]$?

$$Z = b_T v^T = 10000 (1.10)^T e^{-.05T} = 10000 \left(1.10 e^{-.05}\right)^T$$

$$E[Z] = \int_0^{60} 10000 \, (1.1e^{-.05})^t \frac{1}{60} \, dt = \frac{10000}{60} \int_0^{60} (1.1e^{-.05})^t \, dt$$

$$\int a^t dt = \int e^{t(\log a)} = \frac{1}{\log(a)} a^t + K \qquad = \underbrace{\frac{10000}{60} \frac{1}{\log(1.1e^{-.05})} (1.1e^{-.05})^t \Big|_0^{60}}_{52{,}082.66}$$

$$\int e^t dt = e^t$$

$$E[Z^2] = \frac{(10000)^2}{60} \int_0^{60} \left[(1.1e^{-.05})^2\right]^t dt = 4{,}208{,}083{,}171$$

$$Var[Z] = E[Z^2] - (E[Z])^2 = 1{,}495{,}479{,}672$$

$$Pr[Z \le \alpha] = \cancel{Pr[T \ge a]} \quad Pr[T \le a]$$

$$\downarrow$$

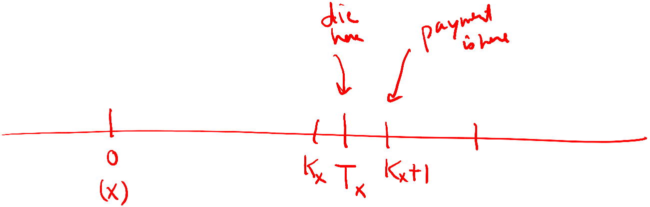
$$10000\,(1.1e^{-.05})^T$$

$$10{,}000\,(1.1e^{-.05})^T \le \alpha$$

$$T \log(1.1e^{-.05}) \le \log\left(\frac{\alpha}{10{,}000}\right)$$

$$T \le \frac{\log(\alpha/10{,}000)}{\log(\underbrace{1.1e^{-.05}}_{>1})}$$

Because ↗ payments, smaller benefit paid means earlier death!



Discrete Insurance

life table =)

Insurances payable at EOY of death

- For insurances payable at the end of the year (EOY) of death, the PV r.v. Z clearly depends on the curtate future lifetime K_x.
- It is $Z = b_{K+1} v_{K+1}$.
- To illustrate, consider an n-year term insurance which pays benefit at the end of year of death:

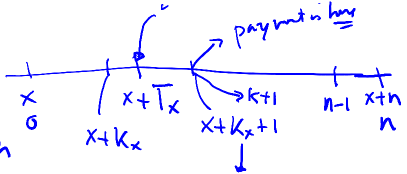
$$b_{K+1} = \begin{cases} 1, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}, \; v_{K+1} = v^{K+1},$$

and therefore

$$Z = \begin{cases} v^{K+1}, & K = 0, 1, \ldots, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

(x)

discrete, in particular b is paid at eoy of death



$K_x = K$

n-year term

$b_{K+1} = 1$

v^{K+1}

$\Pr(K=k) = {}_{k|}q_x$

$= {}_k p_x \, q_{x+k}$

$= \dfrac{l_{x+k}}{l_x} \dfrac{d_{x+k}}{l_{x+k}}$

$$Z = \begin{cases} b_{K+1} v^{K+1}, & K = 0, 1, 2, \ldots, n-1 \\ 0, & K \geq n \end{cases}$$

$$\underbrace{E[Z]}_{\text{APV of an n-year term}} = \sum_{k=0}^{n-1} \cancel{b_{k+1}} v^{k+1} {}_{k|}q_x = \sum_{k=0}^{n-1} \cancel{b_{k+1}} v^{k+1} \frac{d_{x+k}}{l_x}$$

$= \dfrac{d_{x+k}}{l_x}$

$A^1_{x:\overline{n}|}$

$E[Z^2] =$ use rule of moments @ 2δ $= {}^2A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} (v^{k+1})^2 \dfrac{d_{x+k}}{l_x}$

$\mathrm{Var}[Z] = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2$

- continued

$v = \frac{1}{1+i}$

$v^2 = \left(\frac{1}{1+i}\right)^2 = \frac{1}{1+i^*} \Rightarrow i^* = (1+i)^2 - 1$

$\Downarrow$

$@ 2\delta$

- APV of n-year term:

$$A^{1}_{x:\overline{n|}} = \mathsf{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1} {}_{k}p_x \cdot q_{x+k}$$

- Rule of moments also apply in discrete situations. For example,

$$\mathsf{Var}[Z] = {}^{2}A^{1}_{x:\overline{n|}} - \left(A^{1}_{x:\overline{n|}}\right)^2,$$

where

$${}^{2}A^{1}_{x:\overline{n|}} = \mathsf{E}\left[Z^2\right] = \sum_{k=0}^{n-1} e^{-2\delta(k+1)} {}_{k}p_x \cdot q_{x+k}.$$

$@ 2\delta$ $\quad v^2 = e^{-2\delta}$ $\neq$

$n \to \infty$ whole life $Z = v^{K+1}, \; K = 0, 1, \ldots$

$$E[Z] = A_x$$

$$Var[Z] = {}^2A_x - (A_x)^2$$

m-year deferred

$$Z = \begin{cases} 0, & K = 0, \ldots, m-1 \\ v^{K+1}, & K = m, \ldots, \infty \end{cases}$$

$$E[Z] = {}_{m|}A_x$$

$$Var[Z] = {}^2_{m|}A_x - ({}_{m|}A_x)^2$$

n-year endowment

pure endowment + term

$$Z = \begin{cases} v^{K+1}, & K = 0, \ldots, n-1 \\ v^n, & K = n, n+1, \ldots, \infty \end{cases}$$

$$E[Z] = A_{x:\overline{n|}} = A^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}$$

$$Var[Z] = {}^2A_{x:\overline{n|}} - (A_{x:\overline{n|}})^2$$

(Discrete) whole life insurance

Consider a whole life insurance which pays benefit at the end of year of death (for life):

$$b_{K+1} = 1,\ v_{K+1} = v^{K+1}, \text{ and } Z = v^{K+1}.$$

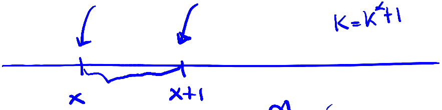
- APV: $A_x = \mathsf{E}[Z] = \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1}\, {}_kq_x \cdot q_{x+k}$
- Applying rule of moments,

  $$\mathsf{Var}[Z] = {}^2A_x - (A_x)^2\,,$$

  where

  $${}^2A_x = \mathsf{E}\left[Z^2\right] = \sum_{k=0}^{\infty} e^{-2\delta(k+1)}\, {}_kp_x \cdot q_{x+k}.$$

Recursion formula



$K = K^* + 1$

whole life insurance

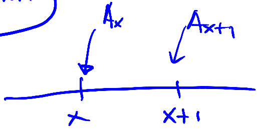
$$A_x = \sum_{k=0}^{\infty} v^{k+1} \underbrace{{}_{k|}q_x}_{{}_kp_x\, q_{x+k}} = v \cdot q_x + \underbrace{\sum_{k=1}^{\infty} v^{k+1}\, {}_kp_x\, q_{x+k}}_{k^* = k-1}$$

$$v = \frac{1}{1+i} = e^{-\delta}$$

$$v\, {}_1p_x \underbrace{\sum_{k^*=0}^{\infty} v^{k^*+1}\, {}_{k^*}p_{x+1}\, q_{x+1+k^*}}_{A_{x+1}}$$

$$\boxed{A_x = \overbrace{v \cdot q_x}^{A^1_{x:\overline{1|}}} + v \cdot p_x\, A_{x+1}}$$

$$= v\, q_x + {}_1E_x\, A_{x+1}$$



$$A_x = A^1_{x:\overline{n|}} + {}_{n|}A_x$$

$$= A^1_{x:\overline{n|}} + \underbrace{v^n\, {}_np_x}_{{}_nE_x}\, A_{x+n}$$

a whole life insurance to (50)

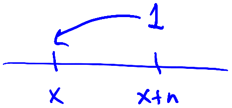
benefit = 100

$$Z = 100\, v^{K+1}, \quad K = 0, 1, \ldots$$

$$E[Z] = 100\, A_{50} = 100 \cdot .24905 = 24.905$$

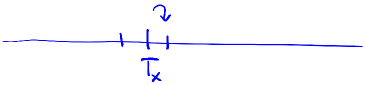
$$Var[Z] = 100^2 \left[{}^2A_{50} - (A_{50})^2 \right]$$

$$= \underbrace{100^2 \left[.09476 - (.24905)^2 \right]}_{\text{calculate this!}}$$



$$v^n \cdot {}_np_x = {}_nE_x$$

discount with life!

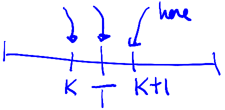


life insurance

continuum $\rightarrow$ discrete

$\int$ moment of death

$\sum$ life table



whole life (discrete)

$b_{K+1} = 1 \qquad v_{K+1} = v^{K+1}, \quad K = 0, 1, 2, \ldots \infty$

A_x

$Z = \text{PV random variable} = b \cdot v = v^{K+1}$

$_{k|}q_x = {}_kp_x \, q_{x+k} = \dfrac{d_{x+k}}{l_x}$

actuarial PV $= E[Z] = E[v^{K+1}] = \sum_{k=0}^{\infty} v^{k+1} P(K=k)$

$E[Z^2] = E[(v^{K+1})^2] = E[e^{-2\delta(K+1)}] = {}^2A_x$

$\text{Var}[Z] = {}^2A_x - (A_x)^2$

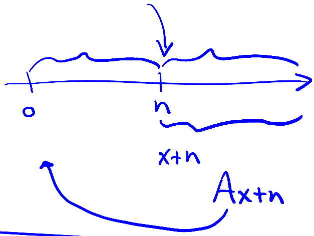
term until n years

deferred starts after n years

$$A^{1}_{x:\overline{n}|} + \underbrace{A_{x:\overline{n}|}^{\;\;1}}_{{}_nE_x} = A_{x:\overline{n}|}$$

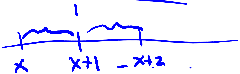
$${}_{n|}A_x = {}_nE_x \, A_{x+n}$$

$$A_x = A^{1}_{x:\overline{n}|} + {}_{n|}A_x$$



Recursive equation:

$$\underbrace{A_x = vq_x + vp_x A_{x+1}}$$



useful for calculating a series of this!

$${}_1E_x = v^1 {}_1p_x = vp_x$$

$${}_nE_x = v^n {}_np_x$$

$$A_x = vq_x + v^2 p_x q_{x+1} + v^2 {}_2p_x A_{x+2}$$

(Discrete) endowment life insurance

- The APV of a (discrete) endowment life insurance is the sum of the APV of a (discrete) term and a pure endowment:

$$A_{x:\overline{n|}} = A^{1}_{x:\overline{n|}} + A_{x:\overset{\;1}{\overline{n|}}}$$

- The policy pays a death benefit of \$1 at the end of the year of death, if death is prior to the end of n years, and a benefit of \$1 if the insured survives at least n years.

- In effect, we have $b_{K+1} = 1$ and $v_{K+1} = \begin{cases} v^{K+1}, & K \leq n-1 \\ v^n, & K \geq n \end{cases}$ so that the PV r.v. is $Z = \begin{cases} v^{K+1}, & K \leq n-1 \\ v^n, & K \geq n \end{cases}$.

- Here $Z = v^{\min(K+1,n)}$ and one can also apply the rule of moments to evaluate the corresponding variance.

Recursive relationships

- The following will be derived/discussed in class:
  - whole life insurance: $A_x = vq_x + vp_x \, A_{x+1}$
  - term insurance: $A^{1}_{x:\overline{n}|} = vq_x + vp_x \, A^{\;\;1}_{x+1:\underline{\underline{\overline{n-1}|}}}$
  - endowment insurance: $A_{x:\overline{n}|} = vq_x + vp_x \, A_{x+1:\underline{\underline{\overline{n-1}|}}}$

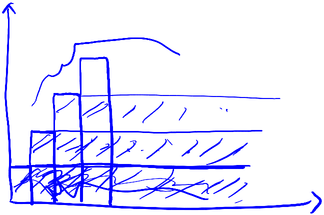
because only n-1 years left

$b_{K+1} = K+1$

v^{K+1}

$Z = (K+1)\,v^{K+1}$

increasing insurance (discrete)



$$E[Z] = E\left[(K+1)\,v^{K+1}\right]$$

$$= \sum_{k=0}^{\infty} (k+1)\,v^{k+1}\;{}_{k|}q_x$$

$(IA)_x \rightarrow$ issue age

increasing insurance

$$(IA)_x = v\,q_x + \underbrace{v\,p_x\,(IA)_{x+1} + v\,p_x\,A_{x+1}}_{+\,v\,p_x\left[(IA)_{x+1} + A_{x+1}\right]}$$

$$\underbrace{A_x + v\,p_x\,A_{x+1} + v^2\,{}_2p_x\,A_{x+2} + \cdots}_{\sum_{k=0}^{\infty} v^k\,{}_kp_x\,A_{x+k}}$$

Illustrative Life Table

Verify:

$x = 50$

$A_{50} = .249\cancel{x}05$

$A_{49} = .23882$

$\to v\,q_{49} + v\,p_{49}\,A_{50}$

$$\underbrace{\frac{1}{1+.06}\left(\frac{5.46}{1000}\right) + \frac{1}{1.06}\left(1-\frac{5.46}{1000}\right)(.24905)}$$

$A_x = v\,q_x + v\,p_x\,A_{x+1}$

$$\frac{A_x - v\,q_x}{v\,p_x} = A_{x+1}$$

${}_nE_x = v^n\,{}_np_x$

$= v^n \dfrac{l_{x+n}}{l_x}$

${}_5E_x \quad {}_{10}E_x \quad {}_{\cancel{20}}E_x$

$A^{1}_{50:\overline{10}|} = A_{50} - {}_{10}E_{50}\,A_{60}$

$A^{1}_{50:\overline{8}|} = A_{50} - \underbrace{{}_8E_{50}}\,A_{58}$

$v^8 \dfrac{l_{58}}{l_{50}} \longrightarrow$ first principle interest rate

risk

MFE

FM

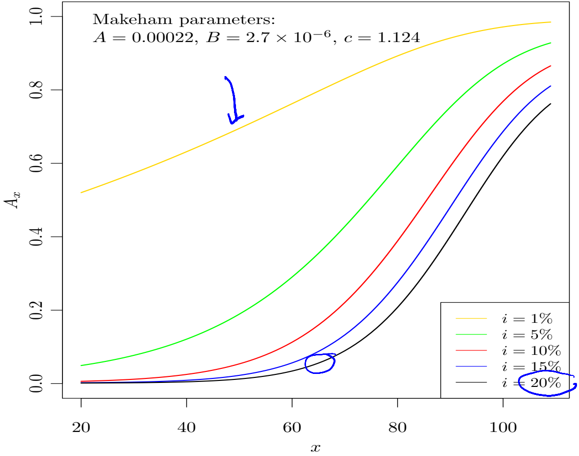


Figure : Actuarial Present Value of a discrete whole life insurance for various interest rate assumptions mortality risk

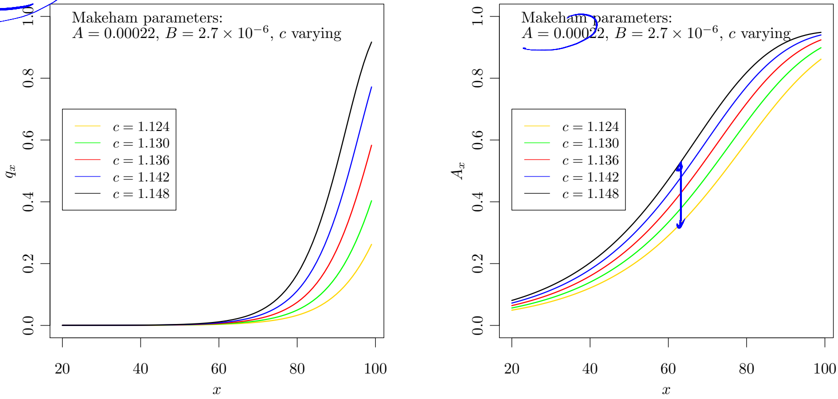


Figure : Actuarial Present Value of a discrete whole life insurance for various mortality rate assumptions with interest rate fixed at 5%

Illustrative example 3

$A_{40} = \frac{1}{1.05}(1-.9972) + \frac{1}{1.05} .9972 A_{41}$

For a whole life insurance of 1 on (41) with death benefit payable at the end of the year of death, let Z be the present value random variable for this insurance.

$A_{40} = v q_{40} + v p_{40} A_{41}$

You are given:

- $i = 0.05$;
- $p_{40} = 0.9972$;
- $A_{41} - A_{40} = 0.00822$; and $\Rightarrow$
- ${}^2A_{41} - {}^2A_{40} = 0.00433$.

${}^2A_{40} = v^2 q_{40} + v^2 p_{40} \, {}^2A_{41}$

$= \left(\frac{1}{1.05}\right)^2 (1-.9972) + \left(\frac{1}{1.05}\right)^2 .9972 \, {}^2A_{41}$

$\left(1 - \frac{1}{1.05} .9972\right) A_{41} = .00822 + \frac{1}{1.05}(1-.9972)$

$\underbrace{\dfrac{.00822 + \frac{1}{1.05}(1-.9972)}{1 - \frac{1}{1.05} .9972}}$

Calculate Var$[Z]$. $= {}^2A_{41} - (A_{41})^2$

$A_{40} = .21699621$

${}^2A_{41} = .0719\,2616$

$$\mathrm{Var}[Z] = .07192616 - (.21699621)^2 \simeq .025$$

3-year discrete term insurance of \$10,000 on (40)

life

$i = 6\%$

$\mu = \mu^{ILT} + .02$

ILT = Illustrative Life Table

Calculate APV of this insurance

$$= 10{,}000 \left[v\, q_{40} + v^2\, p_{40}\, q_{41} + v^3\, p_{40}\, p_{41}\, q_{42} \right]$$

$$p_x = e^{-\int_0^1 \mu_{x+s}\, ds}$$

$$= e^{-\int_0^1 (\mu^{ILT}_{x+s} + .02)\, ds} = p_x^{ILT} e^{-.02}$$

$p_{40} = p_{40}^{ILT} e^{-.02}$

$\left(1 - \frac{2.78}{1000}\right) e^{-.02}$

$p_{41} = \left(1 - \frac{2.98}{1000}\right) e^{-.02}$

$p_{42} = \left(1 - \frac{3.20}{1000}\right) e^{-.02}$

$q_{40} = .0225262\,8$ $\quad q_{41} = .02272232$ $\quad q_{42} = .02293796$

Other forms of insurance

- Deferred insurances
- Varying benefit insurances
- Very similar to the continuous cases
- You are expected to read and understand these other forms of insurances.
- It is also useful to understand the various (possible) recursion relations resulting from these various forms.

Illustration of varying benefits

For a special life insurance issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is \$100,000 in the in the first 10 years of death, decreasing to \$50,000 after that until reaching age 65.
- An endowment benefit of \$100,000 is paid if the insured reaches age 65.
- There are no benefits to be paid past the age of 65.
- Mortality follows the `Illustrative Life Table` at $i = 6\%$.

Calculate the actuarial present value (APV) for this insurance.

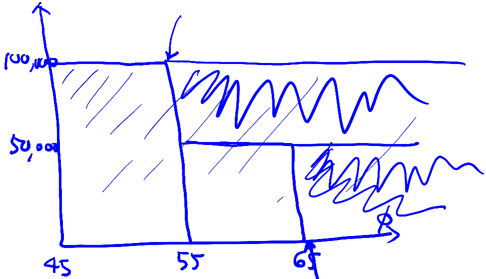
APV(benefits)

$$= 100{,}000\, A_{45} \quad (.20120)$$

$$-\ 50{,}000\ {}_{10}E_{45}\, A_{55} \quad ({}_{10}E_{45} = .52652,\ A_{55} = .30514)$$

$$-\ 50{,}000\ {}_{20}E_{45}\, A_{65} + 100{,}000\ {}_{20}E_{45} \quad ({}_{20}E_{45} = .25634,\ A_{65} = .43980)$$

$$= \boxed{32{,}088.85}$$



1% →

Illustrative example 4

For a whole life insurance issued to age 40, you are given:

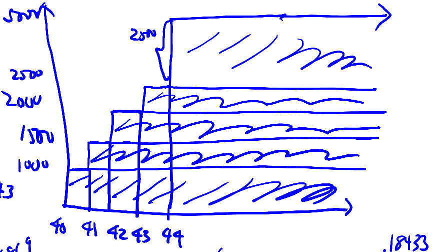
- Death benefits are payable at the moment of death.
- The benefit amount is \$1,000 in the first year of death, increasing by \$500 each year thereafter for the next 3 years, and then becomes level at \$5,000 thereafter.
- Mortality follows the `Illustrative Life Table` at $i = 6\%$.
- Deaths are uniformly distributed over each year of age.

Calculate the APV for this insurance.

Illustrative Ex 4

ILT @ 6%

$${}_nE_x = v^n\,{}_np_x = v^n \frac{l_{x+n}}{l_x}$$



$$APV(\text{policy}) = 1000\,\bar{A}_{40} + 500\,{}_1E_{40}\,\bar{A}_{41} + 500\,{}_2E_{40}\,\bar{A}_{42} + 500\,{}_3E_{40}\,\bar{A}_{43} + 2500\,{}_4E_{40}\,\bar{A}_{44}$$

$i = 6\%$

$\delta = \ln(1.06)$

$l_{41} = 9287264$

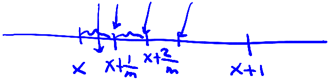
$l_{40} = 9313166$

$$= 500\Big[\,2\,\frac{i}{\delta}\underbrace{A_{40}}_{.16132} + v\,\frac{l_{41}}{l_{40}}\,\frac{i}{\delta}\,A_{41}^{.16869} + v^2\,\frac{l_{42}}{l_{40}}\,\frac{i}{\delta}\,A_{42}^{.17636} + v^3\,\frac{l_{43}}{l_{40}}\,\frac{i}{\delta}\,A_{43}^{.18433}$$

$$+\, v^4\,\frac{l_{44}}{l_{40}}\,\frac{i}{\delta}\,\underbrace{A_{44}}_{.19261}\Big]$$

$$\simeq 7863429$$

Insurances payable m-thly



- Consider the case where we have just one-year term and the benefit is payable at the end of the m-th of the year of death.
- We thus have

$$A_{\overset{1}{x}:\overline{1}|}^{(m)} = \sum_{r=0}^{m-1} v^{(r+1)/m} \cdot {}_{r/m}p_x \cdot {}_{1/m}q_{x+r/m}.$$

- We can show that under the UDD assumption, this leads us to:

$$A_{\overset{1}{x}:\overline{1}|}^{(m)} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{1}|}.$$

- In general, we can generalize this to:

$$A_{\overset{1}{x}:\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}.$$

$$A^{1\,(m)}_{x:\overline{n}|} = A^{1\,(m)}_{x:\overline{1}|} + {}_1E_x \underbrace{A^{1\,(m)}_{x+1:\overline{1}|}} + {}_2E_x A^{1\,(m)}_{x+2:\overline{1}|}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^m = e^\delta$$

$$(1+i)$$

$$\underbrace{\frac{i}{i^{(m)}}} A^1_{x:\overline{1}|} + {}_1E_x \underbrace{\frac{i}{i^{(m)}}} A^1_{x+1:\overline{1}|} + \ldots$$

$$= \frac{i}{i^{(m)}} A^1_{x:\overline{n}|}$$

based on UDD each year approximation

$$\lim_{m\to\infty} i^{(m)} = \delta$$

$n \to \infty$

$$\underbrace{A^{(m)}_x = \frac{i}{i^{(m)}} A_x}$$

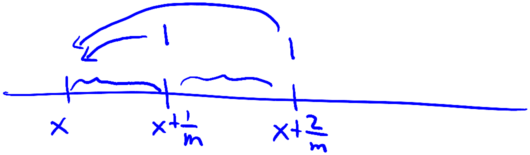
$m \to \infty$

$$\bar{A}_x = \lim_{m\to\infty} A^{(m)}_x = \lim_{m\to\infty} \frac{i}{i^{(m)}} A_x = \frac{i}{\delta} A_x$$

$$\bar{A}_x > A^{(m)}_x > A_x$$

rank of three!

$\underline{a^{(m)}}$



$$\underbrace{v^{\frac{1}{m}}\,{}_{\frac{1}{m}}q_x + v^{\frac{2}{m}}\,{}_{\frac{1}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{1}{m}} + \cdots}$$

$$A^{1\,(m)}_{x:\overline{1}|} = \sum_{r=0}^{m-1} v^{\frac{r+1}{m}}\, \underbrace{{}_{\frac{r}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{r}{m}}}_{\frac{1}{m}\cdot q_x} = \frac{1}{m} q_x \underbrace{\sum_{r=0}^{m-1} v^{(r+1)/m}}_{\underbrace{v^{1/m}+\ldots+v^{m/m}}_{\frac{v^{1/m}\left(1-v^{m/m}\right)}{1-v^{1/m}}}}$$

assume UDD

$$= \frac{i}{i^{(m)}}\left(\overbrace{v q_x}^{A^{1}_{x:\overline{1}|}}\right)$$

Other types of insurances with m-thly payments

- For other types, we can also similarly derive the following (with the UDD assumption):
  - whole life insurance: $A_x^{(m)} = \dfrac{i}{i^{(m)}} A_x$
  - deferred life insurance: ${}_{n|}A_x^{(m)} = \dfrac{i}{i^{(m)}} {}_{n|}A_x$
  - endowment insurance: $A_{x:\overline{n}|}^{(m)} = \dfrac{i}{i^{(m)}} A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$

no adjustment on pure endowment

Relationships - continuous and discrete

$\approx \frac{i}{\delta} A^{1}_{x:\overline{n}|}$

endowment insurance: $\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$ none

- For some forms of insurances, we can get explicit relationships under the UDD assumption:
  - whole life insurance: $\bar{A}_x = \dfrac{i}{\delta} A_x$
  - term insurance: $\bar{A}^{1}_{x:\overline{n}|} = \dfrac{i}{\delta} A^{1}_{x:\overline{n}|}$
  - increasing term insurance: $(I\bar{A})^{1}_{x:\overline{n}|} = \dfrac{i}{\delta}(IA)^{1}_{x:\overline{n}|}$

increasing by 1 each year

at moment of death

Illustrative example 5

select

For a three-year term insurance of 1000 on $[50]$, you are given:

- Death benefits are payable at the end of the quarter of death. m=4
- Mortality follows a select and ultimate life table with a two-year select period:

19 26

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	9706	9687	9661	52
51	9680	9660	9630	53
52	9653	9629	9596	54

31

- Deaths are uniformly distributed over each year of age.
- $i = 5\%$

Calculate the APV for this insurance.

$$APV = 1000 \, A^{(4)\,1}_{[50]:\overline{3}|}$$ quarter

[50] select, $\overline{3}|$ 3 year term

$i = .05$

$i^{(4)} = 4\left[1.05^{1/4} - 1\right] = .049088944$

$$= 1000 \, \frac{i}{i^{(4)}} \, A^{1}_{[50]:\overline{3}|}$$ because of UDD

$$= \underbrace{1000 \, \frac{.05}{.049088944} \, \frac{1}{9706} \left[v(19) + v^2(26) + v^3(31) \right]}_{7.183958}$$

7.183958 3 year term

Illustrative example 6

Use CLT $\sum Z_i \sim$ Normal approximately

amount

Each of 100 independent lives purchases a single premium 5-year deferred whole life insurance of 10 payable at the moment of death.

You are given:

- $\mu = 0.004$

  let Z_i = PV of 5 year deferred of 10

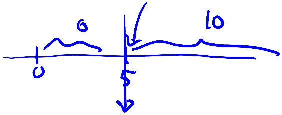
- $\delta = 0.006$
- F is the aggregate amount the insurer receives from the 100 lives.
- The 95th percentile of the standard Normal distribution is 1.645.

Using a Normal approximation, calculate F such that the probability the insurer has sufficient funds to pay all claims is 0.95.

$\mu = .004$

$\delta = .006$

Recall: $\bar{A}_x = \mu / \mu+\delta$



$$E[Z_i] = 10 * e^{-5(\mu+\delta)} \frac{\mu}{\mu+\delta}$$

Let $S = \sum Z_i$

$$Var[Z_i] = 10^2 * \left[e^{-5(\mu+2\delta)} \frac{\mu}{\mu+2\delta} - \left(e^{-5(\mu+\delta)} \frac{\mu}{\mu+\delta} \right)^2 \right]$$

For 100 lives $\Rightarrow$ $E\left[\sum_1^{100} Z_i\right] = 100\, E[Z_i] = 380.5 \longrightarrow$ 3.805 per policy covers you only 1/2 probability approximately

$$Var\left[\sum_1^{100} Z_i\right] = 100\, Var[Z_i] = 860$$

So find F so that

$Pr[F \geq S] = 0.95$ $\Rightarrow$ $Pr\left[\frac{S - E[\sum Z_i]}{\sqrt{Var[\sum Z_i]}} \leq \frac{F - 380.5}{\sqrt{860}} \right] = 0.95$

or $S \leq F$

$Z \sim N(0,1)$

$= 1.645$

In effect, $\dfrac{F - 380.5}{\sqrt{860}} = 1.645$

or $F = \underbrace{380.5 + 1.645\sqrt{860}}_{\text{Mean + percent of S.D.}} = 428.74$

for 100 policies

each policy pays 4.2874

to cover claims

95% of the

time

Illustrative example 7

Suppose interest rate $i = 6\%$ and mortality is based on the following life table:

60 60 60 60 60 60 60 60 60 220 100

x	90	91	92	93	94	95	96	97	98	99	100
ℓ_x	800	740	680	620	560	500	440	380	320	100	0

Calculate the following:

$$A_{94} = \underbrace{\frac{1}{560}\left[v(60) + v^2(60) + v^3(60) + v^4(60) + v^5(220) + (100)v^6 \right]}_{= .790728} < 1$$

(a) A_{94} whole life to (94)

(b) $A^{\;1}_{90:\overline{5}|}$

(c) $_{3|}A^{(4)}_{92}$, assuming UDD between integral ages

(d) $A_{95:\overline{3}|}$

$$A^{1}_{90:\overline{5}|} = \frac{1}{800}\cdot 60\,\underbrace{(v+v^2+v^3+v^4+v^5)}_{\frac{1-v^5}{i}} = .1367993$$

discount to age 92

$${}_{3|}A^{(4)}_{92} = \frac{i}{i^{(4)}}\,{}_{3|}A_{92} = \frac{.06}{4[(1.06)^{1/4}-1]}\cdot\frac{1}{680}\underbrace{\left[v^3(60)+v^4(60)+v^5(60)+v^6(220)+v^7(100)\right]}$$

$i=.06$

$i^{(4)}=4[1.06^{1/4}-1]$

$$= \underline{.5166944}$$

most expensive endowment

$$A_{95:\overline{3}|} = A^{1}_{95:\overline{3}|} + A_{95:\overset{1}{\overline{3}|}}$$

$$= \frac{60}{500}(v+v^2+v^3) + v^3\,\frac{l_{98}}{l_{95}}\ \begin{matrix}\leftarrow 320\\ \leftarrow 500\end{matrix} = .8581178 < 1$$

Illustrative example 8

discrete

A five-year term insurance policy is issued to (45) with benefit amount of \$10,000 payable at the end of the year of death.

Mortality is based on the following select and ultimate life table:

177 249 256

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}		$x+3$
45	5282	5105	4856	4600	491	48
46	4753	4524	4322	4109	359	49
47	4242	4111	3948	3750		50
48	3816	3628	3480	3233		51

Calculate the APV for this insurance if $i = 5\%$.

$$\text{APV (insurance)} = 10{,}000 \cdot \frac{1}{5282}\Big[v(177) + v^2(249) + v^3(256) + v^4(491) + v^5(359) \Big]$$

$v = 1/1.05$

$$= 2{,}462{,}698$$

try calculating variance here!

replace $\delta \to 2\delta$

or equivalently $v \to v^2$

$e \to e^{2\delta}$

Other terminologies and notations used

Expression	Other terms/symbols used
Actuarial Present Value (APV)	Expected Present Value (EPV) Net Single Premium (NSP) single benefit premium
basis	assumptions
interest rate (i)	interest per year effective discount rate
benefit amount (b)	sum insured (S) British death benefit
Expected value of Z	$\mathsf{E}(Z)$
Variance of Z	$\mathsf{Var}(Z)$ $V[Z]$ book