Survival Models

Lecture: Weeks 2-3

Chapter summary

- Survival models
  - Age-at-death random variable
  - Time-until-death random variables
  - Force of mortality (or hazard rate function)
  - Some parametric models
    - De Moivre's (Uniform), Exponential, Weibull, Makeham, Gompertz
    - Generalization of De Moivre's
  - Curtate future lifetime
- Chapter 2 (Dickson, Hardy and Waters = DHW)

Age-at-death random variable

- X is the age-at-death random variable; continuous, non-negative
- X is interpreted as the lifetime of a newborn (individual from birth)
- Distribution of X is often described by its survival distribution function (SDF):

$$S_0(x) = \mathsf{Pr}[X > x]$$

  - other term used: survival function
- Properties of the survival function:
  - $S_0(0) = 1$: probability a newborn survives 0 years is 1.
  - $S_0(\infty) = \lim_{x \to \infty} S_0(x) = 0$: all lives eventually die.
  - non-increasing function of x: not possible to have a higher probability of surviving for a longer period.

Cumulative distribution and density functions

- Cumulative distribution function (CDF): $F_0(x) = \mathsf{Pr}[X \leq x]$
    - nondecreasing; $F_0(0) = 0$; and $F_0(\infty) = 1$.
- Clearly we have: $F_0(x) = 1 - S_0(x)$
- Density function: $f_0\left(x\right) = \dfrac{dF_0(x)}{dx} = -\dfrac{dS_0(x)}{dx}$
    - non-negative: $f_0(x) \geq 0$ for any $x \geq 0$
    - in terms of CDF: $F_0(x) = \int_0^x f_0(z)dz$
    - in terms of SDF: $S_0(x) = \int_x^\infty f_0(z)dz$

Force of mortality

- The force of mortality for a newborn at age x:

$$\mu_x = \frac{f_0(x)}{1-F_0(x)} = \frac{f_0(x)}{S_0(x)} = -\frac{1}{S_0(x)}\frac{dS_0(x)}{dx} = -\frac{d\log S_0(x)}{dx}$$

- Interpreted as the conditional instantaneous measure of death at x.
- For very small Δx, $\mu_x \Delta x$ can be interpreted as the probability that a newborn who has attained age x dies between x and $x+\Delta x$:

$$\mu_x \Delta x \approx \mathsf{Pr}[x < X \leq x + \Delta x | X > x]$$

- Other term used: hazard rate at age x.

Some properties of μ_x

Some important properties of the force of mortality:

- non-negative: $\mu_x \geq 0$ for every $x > 0$
- divergence: $\int_0^\infty \mu_x dx = \infty$.
- in terms of SDF: $S_0(x) = \exp\left(-\int_0^x \mu_z dz\right)$.
- in terms of PDF: $f_0(x) = \mu_x \exp\left(-\int_0^x \mu_z dz\right)$.

Moments of age-at-death random variable

- The mean of X is called the complete expectation of life at birth:

$$\mathring{e}_0 = \mathsf{E}[X] = \int_0^\infty x f_0(x)\, dx = \int_0^\infty S_0(x)\, dx.$$

- The RHS of the equation can be derived using integration by parts.
- Variance:

$$\mathsf{Var}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \mathsf{E}\left[X^2\right] - (\mathring{e}_0)^2.$$

- The median age-at-death m is the solution to

$$S_0(m) = F_0(m) = \frac{1}{2}.$$

Some special parametric laws of mortality

Law/distribution	μ_x	$S_0(x)$	Restrictions
De Moivre (uniform)	$1/(\omega - x)$	$1 - (x/\omega)$	$0 \leq x < \omega$
Constant force (exponential)	μ	$\exp(-\mu x)$	$x \geq 0, \mu > 0$
Gompertz	Bc^x	$\exp\left[-\frac{B}{\log c}(c^x - 1)\right]$	$x \geq 0, B > 0, c > 1$
Makeham	$A + Bc^x$	$\exp\left[-Ax - \frac{B}{\log c}(c^x - 1)\right]$	$x \geq 0, B > 0, c > 1,$ $A \geq -B$
Weibull	kx^n	$\exp\left(-\frac{k}{n+1}x^{n+1}\right)$	$x \geq 0, k > 0, n > 1$

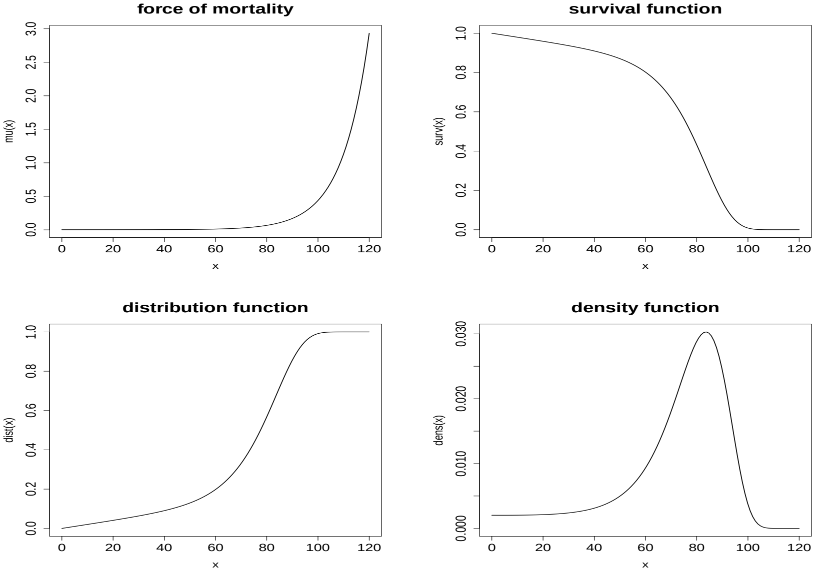

Figure: Makeham's law: $A = 0.002$, $B = 10^{-4.5}$, $c = 1.10$

Illustrative example 1

Suppose X has survival function defined by

$$S_0(x) = \frac{1}{10}(100 - x)^{1/2}, \quad \text{for} \quad 0 \leq x \leq 100.$$

1. Explain why this is a legitimate survival function.
2. Find the corresponding expression for the density of X.
3. Find the corresponding expression for the force of mortality at x.
4. Compute the probability that a newborn with survival function defined above will die between the ages 65 and 75.

Solution to be discussed in lecture.

2.2 Future lifetime random variable

- For a person now age x, its future lifetime is $T_x = X - x$. For a newborn, $x = 0$, so that we have $T_0 = X$.
- Life-age-x is denoted by (x).
- SDF: It refers to the probability that (x) will survive for another t years.

$$S_x(t) = \mathsf{Pr}[T_0 > x + t | T_0 > x] = \frac{S_0(x+t)}{S_0(x)} = {}_tp_x = 1 - {}_tq_x$$

- CDF: It refers to the probability that (x) will die within t years.

$$F_x(t) = \mathsf{Pr}[T_0 \leq x + t | T_0 > x] = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = {}_tq_x$$

- continued

- Density:

$$f_x(t) = \frac{dF_x(t)}{dt} = -\frac{dS_x(t)}{dt} = \frac{f_0(x+t)}{S_0(x)}.$$

- Remark: If $t = 1$, simply use p_x and q_x.
- p_x refers to the probability that (x) survives for another year.
- $q_x = 1 - p_x$, on the other hand, refers to the probability that (x) dies within one year.

2.3 Force of mortality of T_x

- In deriving the force of mortality, we can use the basic definition:

$$\begin{aligned} \mu_x(t) &= \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)}{S_0(x)} \cdot \frac{S_0(x)}{S_0(x+t)} \\ &= \frac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}. \end{aligned}$$

- This is easy to see because the condition of survival to age $x+t$ supercedes the condition of survival to age x.
- This results implies the following very useful formula for evaluating the density of T_x:

$$f_x(t) = {}_tp_x \times \mu_{x+t}$$

Special probability symbol

- The probability that (x) will survive for t years and die within the next u years is denoted by ${}_{t|u}q_x$. This is equivalent to the probability that (x) will die between the ages of $x+t$ and $x+t+u$.
- This can be computed in several ways:

$$\begin{aligned} {}_{t|u}q_x &= \Pr[t < T_x \leq t+u] \\ &= \Pr[T_x \leq t+u] - \Pr[T_x < t] \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x \\ &= {}_tp_x \times {}_uq_{x+t}. \end{aligned}$$

- If $u = 1$, prefix is deleted and simply use ${}_{t|}q_x$.

Other useful formulas

- It is easy to see that

$$F_x(t) = \int_0^t f_x(s)ds$$

which in actuarial notation can be written as

$$_tq_x = \int_0^t {_sp_x}\ \mu_{x+s}ds$$

- See Figure 2.3 for a very nice interpretation.
- We can generalize this to

$$_{t|u}q_x = \int_t^{t+u} {_sp_x}\ \mu_{x+s}ds$$

2.6 Curtate future lifetime

- Curtate future lifetime of (x) is the number of future years completed by (x) prior to death.
- $K_x = \lfloor T_x \rfloor$, the greatest integer of T_x.
- Its probability mass function is

$$\begin{aligned} \mathsf{Pr}[K_x = k] &= \mathsf{Pr}[k \leq T_x < k+1] = \mathsf{Pr}[k < T_x \leq k+1] \\ &= S_x(k) - S_x(k+1) = {}_{k+1}q_x - {}_k q_x = {}_{k|}q_x, \end{aligned}$$

for $k = 0, 1, 2, ...$

- Its distribution function is

$$\mathsf{Pr}[K_x \leq k] = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x.$$

2.5/2.6 Expectation of life

- The expected value of T_x is called the complete expectation of life:

$$\mathring{e}_x = \mathsf{E}[T_x] = \int_0^\infty t f_x(t)dt = \int_0^\infty t\,{}_tp_x \mu_{x+t} dt = \int_0^\infty {}_tp_x dt.$$

- The expected value of K_x is called the curtate expectation of life:

$$e_x = \mathsf{E}[K_x] = \sum_{k=0}^\infty k \cdot \mathsf{Pr}[K_x = k] = \sum_{k=0}^\infty k \cdot {}_{k|}q_x = \sum_{k=1}^\infty {}_kp_x.$$

- Proof can be derived using discrete counterpart of integration by parts (summation by parts). Alternative proof will be provided in class.
- Variances of future lifetime can be similarly defined.

Illustrative Example 2

Let X be the age-at-death random variable with

$$\mu_x = \frac{1}{2(100-x)}, \quad \text{for } 0 \leq x < 100.$$

1. Give an expression for the survival function of X.
2. Find $f_{36}(t)$, the density function of future lifetime of (36).
3. Compute ${}_{20}p_{36}$, the probability that life (36) will survive to reach age 56.
4. Compute $\mathring{e}_{36}$, the average future lifetime of (36).

Illustrative Example 3

Suppose you are given that:

- $\mathring{e}_0 = 30$; and
- $S_0(x) = 1 - \dfrac{x}{\omega}$, for $0 \leq x \leq \omega$.

Evaluate $\mathring{e}_{15}$.

Solution to be discussed in lecture.

Illustrative Example 4

For a group of lives aged 40 consisting of 30% smokers (sm) and the rest, non-smokers (ns), you are given:

- For non-smokers, $\mu_x^{\text{ns}} = 0.05$, for $x \geq 40$
- For smokers, $\mu_x^{\text{sm}} = 0.10$, for $x \geq 40$

Calculate q_{65} for a life randomly selected from those who reach age 65.

Temporary (partial) expectation of life

We can also define temporary (or partial) expectation of life:

$$\mathsf{E}\big[\min(T_x, n)\big] = \mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$$

This can be interpreted as the average future lifetime of (x) within the next n years.

Suppose you are given:

$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x \geq 40 \end{cases}$$

Calculate $\mathring{e}_{25:\overline{25}|}$

Generalized De Moivre's law

The SDF of the so-called Generalized De Moivre's Law is expressed as

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \text{ for } 0 \leq x \leq \omega.$$

Derive the following for this special type of law of mortality:

1. force of mortality
2. survival function associated with T_x
3. expectation of future lifetime of x
4. can you find explicit expression for the variance of T_x?

Illustrative example

- We will do **Example 2.6** in class.

Example 2.3

Let $\mu_x = Bc^x$, for $x > 0$, where B and c are constants such that $0 < B < 1$ and $c > 1$.

Derive an expression for $S_x(t)$.

Typical mortality pattern observed

- High (infant) mortality rate in the first year after birth.
- Average lifetime (nowadays) range between 70-80 - varies from country to country.
- Fewer lives/deaths observed after age 110 - supercentenarian is the term used to refer to someone who has reached age 110 or more.
- The highest recorded age at death (I believe) is 122.
- Different male/female mortality pattern - females are believed to live longer.

Substandard mortality

- A substandard risk is generally referred to someone classified by the insurance company as having a higher chance of dying because of:
  - some physical condition
  - family or personal medical history
  - risky occupation
  - dangerous habits or lifestyle (e.g. skydiving)
- Mortality functions are superscripted with s to denote substandard: q_x^s and μ_x^s.
- For example, substandard mortality may be obtained from a standard table using:
  1. adding a constant to force of mortality: $\mu_x^s = \mu_x + c$
  2. multiplying a fixed constant to probability: $q_x^s = \min(kq_x, 1)$
- The opposite of a substandard risk is preferred risk where someone is classified to have better chance of survival.

Final remark - other contexts

- The notion of a lifetime or survival learned in this chapter can be applied in several other contexts:
  - engineering: lifetime of a machine, lifetime of a lightbulb
  - medical statistics: time-until-death from diagnosis of a disease, survival after surgery
  - finance: time-until-default of credit payment in a bond, time-until-bankruptcy of a company
  - space probe: probability radios installed in space continue to transmit
  - biology: lifetime of an organism
  - other actuarial context: disability, sickness/illness, retirement, unemployment

Other symbols and notations used

Expression	Other symbols used
probability function	$P(\cdot)$ $\Pr(\cdot)$
survival function of newborn	$S_X(x)$ $S(x)$ $s(x)$
future lifetime of x	$T(x)$ T
curtate future lifetime of x	$K(x)$ K
survival function of x	$S_{T_x}(t)$ $S_T(t)$
force of mortality of T_x	$\mu_{T_x}(t)$ $\mu_x(t)$