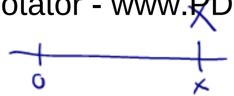
Survival Models

Lecture: Weeks 2-3

Chapter summary

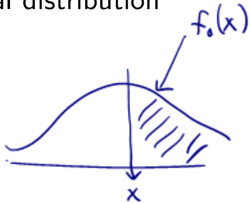
- Survival models
  - Age-at-death random variable
  - Time-until-death random variables
  - Force of mortality (or hazard rate function)
  - Some parametric models
    - De Moivre's (Uniform), Exponential, Weibull, Makeham, Gompertz
    - Generalization of De Moivre's
  - Curtate future lifetime
- Chapter 2 (Dickson, Hardy and Waters = DHW)

Age-at-death random variable



- X is the age-at-death random variable; continuous, non-negative
- X is interpreted as the lifetime of a newborn (individual from birth)
- Distribution of X is often described by its survival distribution function (SDF):

$$S_0(x) = \mathrm{Pr}[X > x]$$



  - other term used: survival function
- Properties of the survival function:
  - $S_0(0) = 1$: probability a newborn survives 0 years is 1.
  - $S_0(\infty) = \lim_{x\to\infty} S_0(x) = 0$: all lives eventually die.
  - non-increasing function of x: not possible to have a higher probability of surviving for a longer period.

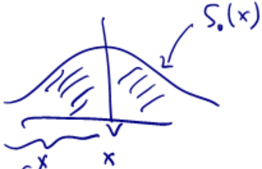
Cumulative distribution and density functions

$= 1 - S_0(x)$

- Cumulative distribution function (CDF): $F_0(x) = \Pr[X \leq x]$
  - nondecreasing; $F_0(0) = 0$; and $F_0(\infty) = 1$.
- Clearly we have: $F_0(x) = 1 - S_0(x)$

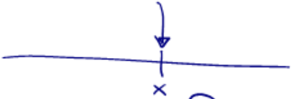
$X =$ continuous ≥ 0

- Density function: $\underline{f_0(x)} = \dfrac{dF_0(x)}{dx} = -\dfrac{dS_0(x)}{dx}$
  - non-negative: $f_0(x) \geq 0$ for any $x \geq 0$
  - in terms of CDF: $F_0(x) = \int_0^x f_0(z)dz$
  - in terms of SDF: $S_0(x) = \int_x^\infty f_0(z)dz$



$\int_0^x f_0(x)dx = F_0(x)$

Force of mortality



- The force of mortality for a newborn at age x:

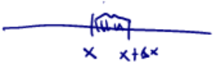
ln

$$\mu_x = \frac{f_0(x)}{1 - F_0(x)} = \frac{f_0(x)}{S_0(x)} = -\frac{1}{S_0(x)} \frac{dS_0(x)}{dx} = -\frac{d \log S_0(x)}{dx}.$$

- Interpreted as the conditional instantaneous measure of death at x.
- For very small Δx, $\mu_x \Delta x$ can be interpreted as the probability that a newborn who has attained age x dies between x and $x + \Delta x$:

$$\mu_x \Delta x \approx \Pr[x < X \leq x + \Delta x | X > x]$$

- Other term used: hazard rate at age x.



$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\underbrace{\Pr[\overset{A}{x<X\le x+\Delta x} \mid \overset{B}{X>x}]}$$

$$\frac{\Pr[x<X\le x+\Delta x,\ X>x]}{\Pr[X>x]}$$

$$\Pr[X>x] \to S_0(x)$$

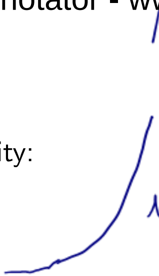
$$\frac{f_0(x)}{S_0(x)}$$

$$= \frac{1}{S_0(x)}\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(S_0(x)-S_0(x+\Delta x)\right)$$

$$= \frac{-1}{S_0(x)}\lim_{\Delta x\to 0}\underbrace{\frac{S_0(x+\Delta x)-S_0(x)}{\Delta x}}_{\frac{dS_0(x)}{dx}}$$

μ_x

Some properties of μ_x



$-\frac{d}{dx}\log S_0(x)$

$\mu_x = \frac{-dS_0(x)}{S_0(x)\,dx}$

Some important properties of the force of mortality:

- non-negative: $\mu_x \geq 0$ for every $x > 0$
- divergence: $\int_0^\infty \mu_x dx = \infty$.
- in terms of SDF: $S_0(x) = \exp\left(-\int_0^x \mu_z dz\right)$.
- in terms of PDF: $f_0(x) = \mu_x \underbrace{\exp\left(-\int_0^x \mu_z dz\right)}_{S_0(x)}$.

$S_0(x)$

$\mu_z = -\frac{d}{dz}\log S_0(z)$

$e^{\int_0^x -\mu_z dz} = e^{\int_0^x d\log S_0(z)}$

$= e^{\log S_0(x)}$.

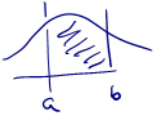
μ_x

$$Pr[a < X < b]$$

$$= \underbrace{Pr[X > a]} - Pr[X > b]$$

$$= \underbrace{e^{-\int_0^a \mu_z dz} - e^{-\int_0^b \mu_z dz}}$$

$=$



$$e^{-\int} = e^{-\int_a^b \mu_z dz}$$

Moments of age-at-death random variable

- The mean of X is called the complete expectation of life at birth:

$$\underset{\text{birth}}{\mathring{e}_0} = \mathsf{E}[X] = \int_0^\infty x f_0(x)\, dx = \int_0^\infty S_0(x)\, dx.$$

- The RHS of the equation can be derived using integration by parts.
- Variance:

$$\mathsf{Var}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \mathsf{E}\left[X^2\right] - (\mathring{e}_0)^2 .$$

- The median age-at-death m is the solution to

$$S_0(m) = F_0(m) = \frac{1}{2}.$$

$$f_0(x) = \frac{dF_0(x)}{dx} = \frac{-dS_0(x)}{dx}$$

$$\underbrace{\int u\,dv = uv - \int v\,du}$$

$$E[x] = \int_0^\infty S_0(x)$$

$$\downarrow$$

$$\int_0^\infty x \underbrace{f_0(x)dx}_{-dS_0(x)} = -\int_0^\infty x \, dS_0(x)$$

$$u \; dv$$

$$u = x, \quad dv = d\underbrace{S_0(x)}_{v}$$

$$= -x S_0(x) \Big|_0^\infty + \int_0^\infty S_0(x)\,dx$$

$$= -\infty S_0(\infty) + \cancel{0 S_0(0)} \to 0$$

$$\downarrow \; 0$$

$S_0(x) \to 0$ faster than x

$$\mathring{e}_0 = \int_0^\infty S_0(x)\,dx$$

Some special parametric laws of mortality

$\frac{f_0(x)}{S_0(x)} \Rightarrow \mu_x S_0(x) = f_0(x)$

Law/distribution	μ_x	$S_0(x)$	$f_0(x)$	Restrictions
* De Moivre (uniform)	$1/(\omega - x)$	$1 - (x/\omega) = \frac{\omega - x}{\omega}$	$\frac{1}{\omega}$	$0 \leq x < \omega$ everyone gone
* Constant force (exponential)	μ	$\exp(-\mu x)$		$x \geq 0, \mu > 0$
Gompertz	Bc^x	$\exp\left[-\frac{B}{\log c}(c^x - 1)\right]$		$x \geq 0, B > 0, c > 1$
Makeham	$A + Bc^x$ accident	$\exp\left[-Ax - \frac{B}{\log c}(c^x - 1)\right]$		$x \geq 0, B > 0, c > 1, A \geq -B$
Weibull	kx^n	$\exp\left(-\frac{k}{n+1}x^{n+1}\right)$		$x \geq 0, k > 0, n > 1$

$\mu_x = Bc^x \qquad c^z = e^{z(\log c)}$

$$S_0(x) = e^{-\int_0^x Bc^z\,dz}$$

$$= e^{-B\int_0^x e^{z(\log c)}\,dz}$$

$$= e^{-\frac{B}{\log c}\,\underbrace{e^{z\log(c)}}_{c^z}\Big|_0^x}$$

$$= e^{-\frac{B}{\log c}(c^x - c^0)}$$

$$= e^{-\frac{B}{\log c}(c^x - 1)}$$

$$f_0(x) = \mu_x S_0(x)$$

$$\mu_x = \frac{1}{\omega - x}$$

$$S_0(x) = e^{-\int_0^x \frac{1}{\omega - z}\,dz} = e^{\log(\omega - z)\Big|_0^x}$$

$$= e^{(\log(\omega - x) - \log(\omega))}$$

$$= \cancel{e^{\log}}\, \frac{\omega - x}{\omega} = \frac{\omega - x}{\omega} = 1 - \frac{x}{\omega}$$

$$f_0(x) = \mu_x S_0(x) = \frac{1}{\omega}$$

Mortality follows de Moivre with ω $\downarrow$ limiting age

Exponential (constant force)

$$\mu_x = \mu$$

$$S_0(x) = e^{-\mu x}$$

$$f_0(x) = \mu_x S_0(x) = \mu e^{-\mu x}, \quad x > 0$$

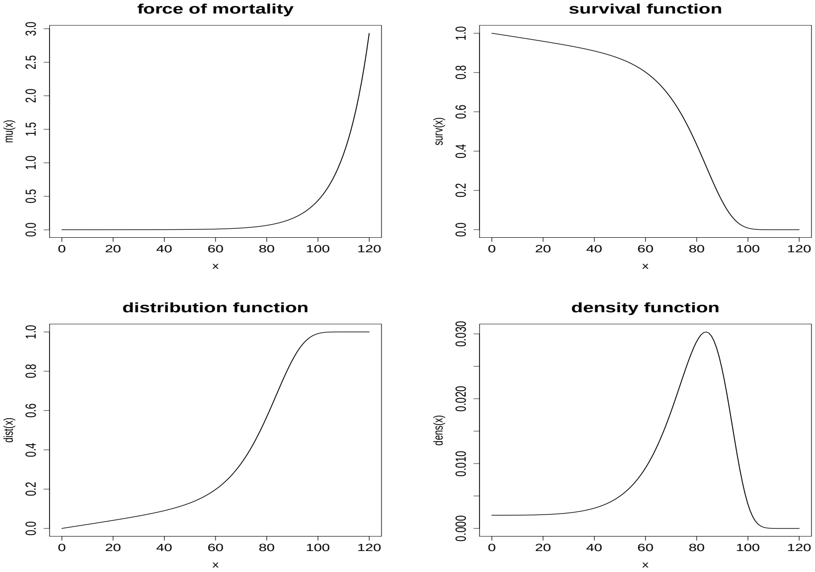


Figure: Makeham's law: $A = 0.002$, $B = 10^{-4.5}$, $c = 1.10$

Illustrative example 1

Generalized de Moivre

Suppose X has survival function defined by

$$S_0(x) = \underbrace{\frac{1}{10}(100-x)^{1/2}}_{\left(\frac{100-x}{100}\right)^{1/2}}, \quad \text{for} \quad 0 \leq x \leq 100.$$

1. Explain why this is a legitimate survival function.
2. Find the corresponding expression for the density of X.
3. Find the corresponding expression for the force of mortality at x.
4. Compute the probability that a newborn with survival function defined above will die between the ages 65 and 75.

Solution to be discussed in lecture.

$Pr[65 < X \leq 75] = Pr[X > 65] - Pr[X > 75]$

$= S_0(65) - S_0(75)$

$.09161 = \frac{1}{10}(\sqrt{35} - \sqrt{25})$

- $S_0(0) = \frac{1}{10}(100)^{1/2} = 1$

①

- $S_0(\infty) = S_0(100) = \frac{1}{10}(100-100)^{1/2} = 0$

- $\underbrace{\text{nonincreasing}}$ $\frac{d}{dx} S_0(x) \leq 0$

$$S_0(x) = \frac{1}{10}(100-x)^{1/2}$$

$$\frac{d}{dx} S_0(x) = \underbrace{\frac{1}{10} \cdot \frac{1}{2}(100-x)^{-1/2}(-1)}_{\leq 0}$$

② $f_0(x) = -\frac{d}{dx} S_0(x) = \frac{1}{20}(100-x)^{-1/2}$

③ $\mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\frac{1}{20}(100-x)^{-1/2}}{\frac{1}{10}(100-x)^{1/2}} = \frac{1}{2} \frac{1}{100-x}$

2.2 Future lifetime random variable

(65) (70)

- For a person now age x, its future lifetime is $T_x = X - x$. For a newborn, $x = 0$, so that we have $T_0 = X$.

  $x=0 \Rightarrow T_0 = X$

- Life-age-x is denoted by (x).

  p = survival
  q = death

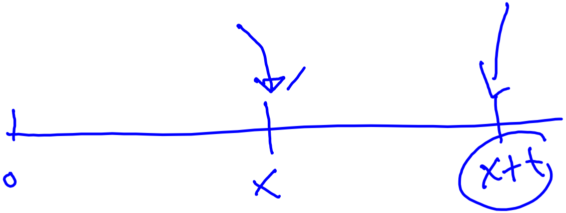
- SDF: It refers to the probability that (x) will survive for another t years.

  $T_0 \geq x+t \cap T_0 > x \Rightarrow$

$$\Pr[T_x > t] = S_x(t) = \Pr[\overbrace{T_0 > x + t}^{A} | \overset{B}{T_0 > x}] = \frac{S_0(x+t)}{S_0(x)} = {}_tp_x = 1 - {}_tq_x$$

- CDF: It refers to the probability that (x) will die within t years.

$$\Pr[T_x \leq t] \quad F_x(t) = \Pr[T_0 \leq x + t | T_0 > x] = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = {}_tq_x$$



$$X \to f_0(x),\ S_0(x),\ \mu_x,\ F_0(x)$$

describes X

$$T_x \to f_x(t),\ S_x(t),\ \mu_x(t),\ F_x(t)$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)}$$

$$F_x(t) = 1 - S_x(t) = \frac{S_0(x) - S_0(x+t)}{S_0(x)}$$

$$f_x(t) = -\frac{d}{dt} S_x(t) = \frac{d}{dt} F_x(t)$$

$$= -\frac{d}{dt}\left(\frac{S_0(x+t)}{S_0(x)}\right)$$

$$= -\frac{1}{S_0(x)} S_0'(x+t) = \frac{f_0(x+t)}{S_0(x)}$$

$$-S' = f$$

$$\mu_x(t) = \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)}{\cancel{S_0(x)}} \Big/ \frac{S_0(x+t)}{\cancel{S_0(x)}} = \frac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}$$

Exponential case

$\mu_x = \mu$, constant
independent of x

$$S_0(x) = e^{-\int_0^x \mu\,dz} = e^{-\mu x}$$

$$f_0(x) = \mu e^{-\mu x}, \quad x \geq 0 \implies \text{Exponential}$$

$${}_tp_x = \Pr[T_x > t] = \frac{S_0(x+t)}{S_0(x)}$$

$${}_tq_x = \Pr[T_x \leq t]$$

$$\mu_{x+t} = \mu_x = \mu$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{e^{-\mu(\cancel{x}+t)}}{\cancel{e^{-\mu x}}} = e^{-\mu t}$$

$$X \sim \text{Exp}(\mu)$$

$$f_x(t) = \mu e^{-\mu t}, \quad t \geq 0 \implies T_x \sim \text{Exp}(\mu)$$

$\cancel{f_x(t)} =$

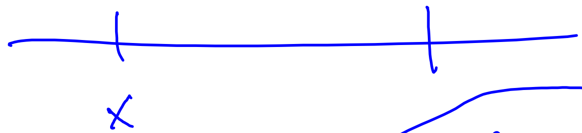
$$\mu_x(t) = \frac{f_x(t)}{S_x(t)}$$

T_x

T_x

$$f_x(t) = S_x(t)\,\mu_x(t)$$

$$= {}_t p_x \, \mu_{x+t}$$



$g(T_x)$

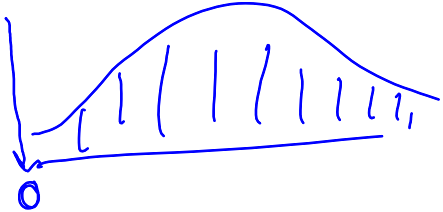
$$\underbrace{\int_0^\infty g(t) f_x(t)\,dt}_{\frac{f_0(x+t)}{S_0(x)}} = E[g(T_x)]$$

$\mathrm{Var}[g(T_x)]$

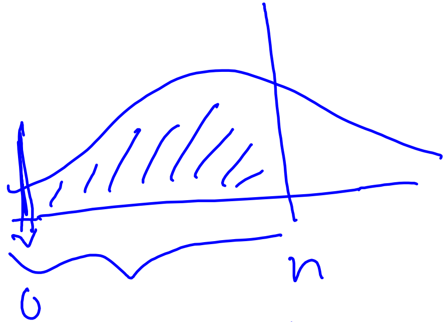
$q = 1 - p$

$p = 1 - q$

$$\int_0^\infty \underbrace{{}_tp_x\,\mu_{x+t}}_{f_x(t)}\,dt = 1.0$$

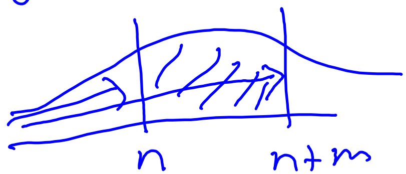


$$ {}_nq_x = \int_0^n {}_tp_x\,\mu_{x+t}\,dt$$



$$\underbrace{{}_{n+m}q_x - {}_nq_x}_{{}_np_x - {}_{n+m}p_x} = \int_n^{n+m} {}_tp_x\,\mu_{x+t}\,dt$$

$$ {}_np_x - {}_{n+m}p_x = {}_{n|m}q_x$$

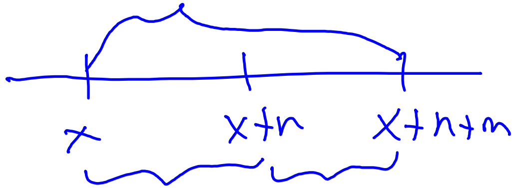


$$_{n|m}q_x = {}_np_x \cdot {}_mq_{x+n}$$

$\underbrace{\text{p's are multiplicative}}$

$$\cancel{_{n+m}q_x \neq {}_nq_x \cdot {}_mq_{x+n}} \Leftarrow \underbrace{\text{q's are } \underline{\text{not}} \text{ multiplicative}}$$

$$_{n+m}q_x = {}_nq_x + {}_np_x \, {}_mq_{x+n}$$



$$_tq_x \Rightarrow t=1 \quad q_x$$

$$_{n+m}p_x = {}_np_x \cdot {}_mp_{x+n}$$

$$p_x \; p_{x+1} \; p_{x+2} \cdots p_{x+n+m-1}$$

$$_1p_x \; {}_1p_{x+1} \; {}_1p_{x+2} \cdots {}_1p_{x+n+m-1} = e^{-\int_0^{n+m} \mu_{x+s} \, ds}$$

$t = 1$ $\quad {}_{1}p_x = p_x$

$${}_{1}q_x = q_x$$

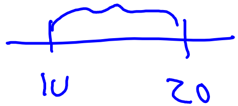
${}_{u|t}q_x$ $\quad {}_{u|1}q_x = {}_{u|}q_x = p_{x+u} - p_{x+u+1}$

$$= q_{x+u+1} - q_{x+u}$$

Survival model: $S_0(x) = \dfrac{1}{x+1}, \quad x \geq 0$

Calculate:

$${}_{10}p_{10}$$

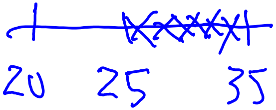


legitimate

- $S_0(0) = 1$
- $S_0(\infty) = 0$
- nonincreasing

$$\frac{d}{dx}(x+1)^{-1} = -(x+1)^{-2}$$

$${}_{5|10}q_{20}$$



$$\mu_{20} \rightarrow$$

$${}_tp_x = \frac{S_0(x+t)}{S_0(x)}$$

$${}_{10}p_{10} = \frac{S_0(20)}{S_0(10)} = \frac{1/21}{1/11} = \frac{11}{21} = ?$$

$$\mu_x = \frac{-dS_0(x)}{S_0(x)\,dx} = \frac{+(x+1)^{-2}}{\cancel{(x+1)}^{-1}} = \frac{1}{x+1} = \frac{1}{21}$$

$${}_{5|10}q_{20} = {}_5p_{20} - {}_{15}p_{20} = \frac{21}{26} - \frac{21}{36} =$$

- continued

- Density:

$$f_x(t) = \frac{dF_x(t)}{dt} = -\frac{dS_x(t)}{dt} = \frac{f_0(x+t)}{S_0(x)}.$$

- Remark: If $t = 1$, simply use p_x and q_x.
- p_x refers to the probability that (x) survives for another year.
- $q_x = 1 - p_x$, on the other hand, refers to the probability that (x) dies within one year.

2.3 Force of mortality of T_x

- In deriving the force of mortality, we can use the basic definition:

$$\begin{aligned} \mu_x(t) &= \frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)}{S_0(x)} \cdot \frac{S_0(x)}{S_0(x+t)} \\ &= \frac{f_0(x+t)}{S_0(x+t)} = \mu_{x+t}. \end{aligned}$$

- This is easy to see because the condition of survival to age $x+t$ supercedes the condition of survival to age x.
- This results implies the following very useful formula for evaluating the density of T_x:

$$f_x(t) = {}_tp_x \times \mu_{x+t}$$

Special probability symbol

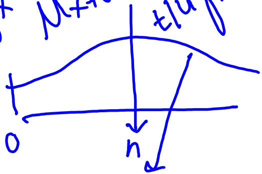
- The probability that (x) will survive for t years and die within the next u years is denoted by ${}_{t|u}q_x$. This is equivalent to the probability that (x) will die between the ages of $x+t$ and $x+t+u$.
- This can be computed in several ways:

$$\begin{aligned}
{}_{t|u}q_x &= \text{Pr}[t < T_x \le t+u] \\
&= \text{Pr}[T_x \le t+u] - \text{Pr}[T_x < t] \\
&= {}_{t+u}q_x - {}_tq_x \\
&= {}_tp_x - {}_{t+u}p_x \\
&= {}_tp_x \times {}_uq_{x+t}.
\end{aligned}$$

- If $u=1$, prefix is deleted and simply use ${}_{t|}q_x$.

Other useful formulas

$_tp_x \quad _tq_x \quad \mu_{x+t} \quad _{t|u}q_x$



- It is easy to see that

$$F_x(t) = \int_0^t f_x(s)ds$$

which in actuarial notation can be written as

$$_tq_x = \int_0^t {}_sp_x \; \mu_{x+s} ds$$

$_tp_x \, \mu_{x+t}$

$$\int_0^n {}_tp_x \, \mu_{x+t} \, dt$$

- See Figure 2.3 for a very nice interpretation.
- We can generalize this to

$$_{t|u}q_x = \int_t^{t+u} {}_sp_x \; \mu_{x+s} ds$$

${}_tp_x$

${}_tq_x$ ${}_{t|u}q_x$

$S_0(x) \to f_0(x)$
$\to \mu_x$
$\to F_0(x)$

$f_0(x) \to S_0(x),\ F_0(x)\ \mu_x$

Given	
$f_0(x)$	$F_0(x)=\int_0^x f_0(z)\,dz$, $S_0(x)=1-\int_0^x f_0(z)\,dz=\int_x^\infty f_0(z)\,dz$, $\mu_x=\dfrac{f_0(x)}{\int_x^\infty f_0(z)\,dz}$
$F_0(x)$	$f_0(x)=\frac{d}{dx}F_0(x)$, $S_0(x)=1-F_0(x)$, $\mu_x=\dfrac{\frac{d}{dx}F_0(x)}{1-F_0(x)}$
$S_0(x)$	$f_0(x)=\frac{-d}{dx}S_0(x)$, $F_0(x)=1-S_0(x)$, $\mu_x=\dfrac{\frac{-d}{dx}S_0(x)}{S_0(x)}=\frac{-d}{dx}\log S_0(x)$
μ_x	$S_0(x)=e^{-\int_0^x \mu_z\,dz}$, $F_0(x)=1-e^{-\int_0^x \mu_z\,dz}$, $f_0(x)=\mu_x\cdot e^{-\int_0^x \mu_z\,dz}$

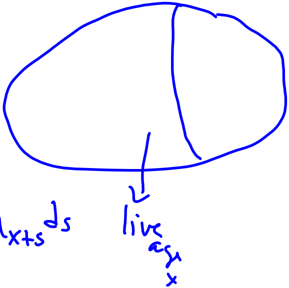
$S_0(x) \quad f_0(x) \quad \mu_x \quad F_0(x)$

To derive the distribution of T_x:

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)}$$

$$\frac{e^{-\int_0^{x+t} \mu_z dz}}{e^{-\int_0^{x} \mu_z dz}} \leftarrow S_x(t) = \frac{S_0(x+t)}{S_0(x)} = {}_tp_x = e^{-\int_x^{x+t} \mu_z dz}$$

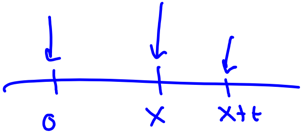
$$= e^{-\int_0^{t} \mu_{x+s} ds}$$



$${}_tq_x = F_x(t) = 1 - S_x(t)$$

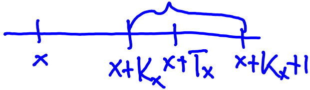
$$\Downarrow$$

$$1 - {}_tp_x \qquad \mu_{x+t} = \mu_x(t)$$



2.6 Curtate future lifetime

Discrete



- Curtate future lifetime of (x) is the number of future years completed by (x) prior to death.
- $K_x = \lfloor T_x \rfloor$, the greatest integer of T_x.

0, 1, 2, 3, ...

- Its probability mass function is

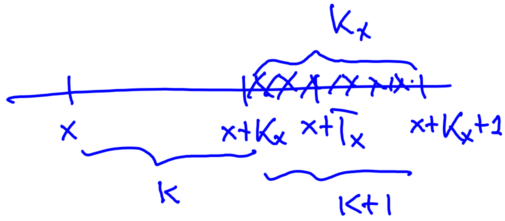
$$\begin{aligned} \Pr[K_x = k] &= \Pr[k \leq T_x < k+1] = \Pr[k < T_x \leq k+1] \\ &= S_x(k) - S_x(k+1) = {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x, \end{aligned}$$

for $k = 0, 1, 2, ...$

- Its distribution function is

$$\Pr[K_x \leq k] = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x.$$

$${}_{k|}q_x$$

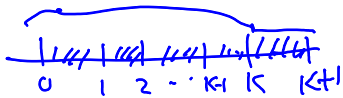


$$\Pr[K_x = k] = \Pr[k < T_x \le k+1]$$

$$= {}_{k|1}q_x = {}_{k|}q_x = {}_{k+1}q_x - {}_{k}q_x = {}_{k}p_x - {}_{k+1}p_x$$

$$= {}_{k}p_x \; q_{x+k}$$

$$\sum_{k=0}^{\infty} \Pr[K_x = k] = 1 = \sum_{k=0}^{\infty} {}_{k|}q_x$$



$$\Pr[K_x \le k] = \sum_{j=0}^{k} \Pr[K_x = j] = \sum_{j=0}^{k} \underbrace{{}_{j|}q_x}_{{}_{j+1}q_x - {}_{j}q_x} = {}_{k+1}q_x$$

$$= \cancel{{}_{1}q_x} + \cancel{{}_{2}q_x} + \cdots + {}_{k+1}q_x - ({}_{0}q_x + \cancel{{}_{1}q_x} + \cdots + \cancel{{}_{k}q_x}) = {}_{k+1}q_x - \underbrace{{}_{0}q_x}_{\substack{\shortparallel \\ 0}}$$

2.5/2.6 Expectation of life

$E(x) = \mathring{e}_0 = \int_0^\infty S_0(x)\,dx$

- The expected value of T_x is called the complete expectation of life: $= \mathring{e}_x$

$$\mathring{e}_x = \mathsf{E}[T_x] = \int_0^\infty t f_x(t)dt = \int_0^\infty t\,{}_tp_x \mu_{x+t} dt = \int_0^\infty {}_tp_x dt.$$

- The expected value of K_x is called the curtate expectation of life: $= e_x$

$$e_x = \mathsf{E}[K_x] = \sum_{k=0}^{\infty} k \cdot \mathsf{Pr}[K_x = k] = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x.$$

- Proof can be derived using discrete counterpart of integration by parts (summation by parts). Alternative proof will be provided in class.

- Variances of future lifetime can be similarly defined.

$e_x < \mathring{e}_x \Rightarrow \mathring{e}_x \approx e_x + 1/2$

$$\mathring{e}_x = \int_0^\infty t \cdot \underbrace{f_x(t)\,dt}_{-d\,S_x(t)}$$

by parts

$u = t \quad dv = d\,S_x(t)$

$v = S_x(t)$

$du = dt$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$-\frac{d}{dt}S_x(t) = f_x(t)$$

$$= -\cancel{t\,S_x(t)}\Big|_0^\infty + \int_0^\infty S_x(t)\,dt = \int_0^\infty \underbrace{S_x(t)}_{{}_tp_x}\,dt = \underline{\int_0^\infty {}_tp_x\,dt}$$

$$\lim_{t\to\infty} t\,S_x(t) = 0 \quad \text{assumption } 2.6$$

$$e_x = \sum_{k=0}^\infty k \cdot \underbrace{{}_{k|}q_x}_{{}_kp_x - {}_{k+1}p_x}$$

$0 \quad k \quad k+1$

$$= \sum_{k=1}^\infty k\left({}_kp_x - {}_{k+1}p_x\right)$$

$$= \sum_{k=1}^\infty k\,{}_kp_x - \sum_{k=1}^\infty k \cdot {}_{k+1}p_x$$

$$= {}_1p_x + 2\,{}_2p_x + 3\,{}_3p_x + \cdots$$

$$-(\quad {}_2p_x + 2\cdot{}_3p_x +$$

$$= {}_1p_x + {}_2p_x + {}_3p_x + \cdots$$

$$e_x = \sum_{k=1}^\infty {}_kp_x$$

Remember:

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt$$

and

$$e_x = \sum_{k=1}^\infty {}_kp_x$$

Example 2.6 of DHW [Notes rewritten!!]

Given $F_0(x) = 1 - \left(1 - x/120\right)^{1/6}$, $0 \le x \le 120$

Evaluate $\mathring{e}_x = E[T_x]$ and $Var[T_x]$ $x=30$ and 80

First evaluate $$ {}_tp_x = \frac{S_0(x+t)}{S_0(x)} = \frac{\left(1 - \frac{x+t}{120}\right)^{1/6}}{\left(1 - x/120\right)^{1/6}} = \left(\frac{120-x-t}{120-x}\right)^{1/6} $$

$$ = \left(1 - t/120-x\right)^{1/6} $$

$$ \mathring{e}_x = \int_0^\infty {}_tp_x\,dt = \int_0^{120-x} \left(1 - t/120-x\right)^{1/6} dt $$

substitution $u = 1 - t/120-x$

$du = \frac{-1}{120-x}\,dt$

$$ = -(120-x)\int_1^0 u^{1/6}\,du = -(120-x)\left.\frac{u^{7/6}}{7/6}\right|_1^0 = \frac{6}{7}(120-x) $$

$$ \mathring{e}_{30} \Rightarrow \frac{6}{7}(120-30) = \frac{6}{7}(90) = \frac{540}{7} = 77.14286 $$

$$Var[T_x] = E[T_x^2] - (E[T_x])^2 = E[T_x^2] - \left(\frac{540}{7}\right)^2$$

need density of T_x

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)}$$

$$S_0(x) = \left(1 - x/120\right)^{1/6}$$

$$f_0(x) = -\frac{d}{dx} S_0(x) = \frac{1}{120}\left(1 - \frac{x}{120}\right)^{-5/6}\left(\frac{1}{6}\right)$$

substitute $x = 30$

$$f_{30}(t) = \frac{\frac{1}{120}\left(1 - \frac{30+t}{120}\right)^{-5/6}\left(\frac{1}{6}\right)}{\left(1 - 30/120\right)^{1/6}} = \frac{\frac{1}{\cancel{120}}\,\cancel{120}^{1/6}\,\cancel{120}^{5/6}\left(\frac{1}{6}\right)(90-t)^{-5/6}}{90^{1/6}} = \frac{1}{6}\frac{1}{90}\left(\frac{90-t}{90}\right)^{-5/6}$$

$$= \frac{1}{540}\left(1 - t/90\right)^{-5/6}$$

$$E[T_{30}^2] = \int_0^{90} t^2 \frac{1}{540}\left(1 - t/90\right)^{-5/6} dt = \frac{1}{\cancel{540}\,6}\int_1^0 90(1-u)^2 u^{-5/6}(-\cancel{90})\,du$$

apply substitution $u = 1 - t/90 \quad du = -\frac{1}{90}dt$

$t = 90(1-u)$

$$= -\frac{1}{6}90^2\left[\int_1^0 (1-2u+u^2)u^{-5/6}du\right] = \frac{90^2}{6}\left[\frac{u^{1/6}}{1/6} - \frac{2u^{7/6}}{7/6} + \frac{u^{13/6}}{13/6}\right]_1^0$$

$$= -\frac{90^2}{\cancel{6}}\cdot\cancel{6}\left(-1 + \frac{2}{7} - \frac{1}{13}\right) = 90^2\left(\frac{72}{91}\right) = \frac{583200}{91}$$

$$\mathrm{Var}[T_{30}] = E[T_{30}^2] - \left(\frac{540}{7}\right)^2 = \frac{583200}{91} - \left(\frac{540}{7}\right)^2$$

$$= \underline{\underline{457.7708}}$$

Illustrative Example 2

Let X be the age-at-death random variable with

$$\mu_x = \frac{1}{2(100-x)}, \quad \text{for } 0 \leq x < 100.$$

1. Give an expression for the survival function of X.
2. Find $f_{36}(t)$, the density function of future lifetime of (36).
3. Compute ${}_{20}p_{36}$, the probability that life (36) will survive to reach age 56.
4. Compute $\mathring{e}_{36}$, the average future lifetime of (36).

Solution to p. 18 Illustrative Example #2

$$\mu_x = \frac{1}{2}\frac{1}{100-x}, \quad 0 \le x < 100$$

① $$S_0(x) = e^{-\int_0^x \mu_z dz} = e^{-\frac{1}{2}\int_0^x \frac{1}{100-z}dz} = e^{\frac{1}{2}\left[\log(100-x) - \log(100)\right]}$$

$$= \left(\frac{100-x}{100}\right)^{1/2}, \quad 0 \le x < 100$$

② $$f_{36}(t) = \frac{f_0(36+t)}{S_0(36)}$$

$$f_0(x) = -\frac{d}{dx}S_0(x) = \frac{1}{2}\frac{1}{100}\left(\frac{100-x}{100}\right)^{-1/2}$$

$$= \frac{\frac{1}{200}\left(\frac{64-t}{100}\right)^{-1/2}}{\left(\frac{64}{100}\right)^{1/2}} = \frac{\frac{1}{200}100^{1/2}(64-t)^{-1/2}}{\left(\frac{64}{100}\right)^{1/2}} = \frac{1}{16}(64-t)^{-1/2}$$

③ $$_{20}p_{36} = \frac{S_0(20+36)}{S_0(36)} = \frac{S_0(56)}{S_0(36)} = \left(\frac{\frac{44}{\cancel{100}}}{\frac{64}{\cancel{100}}}\right)^{1/2} = \left(\frac{11}{16}\right)^{1/2} = .8291562$$

④ $\mathring{e}_{36} = \int_0^{64} {}_tp_{36}\,dt = \int_0^{64} \frac{S_0(36+t)}{S_0(36)}\,dt = \int_0^{64} \left(\frac{\frac{64-t}{\cancel{100}}}{\frac{64}{\cancel{100}}} \right)^{1/2} dt$

$= \frac{1}{8} \int_0^{64} (64-t)^{1/2}\,dt$

$= \frac{-1}{8} \frac{(64-t)^{3/2}}{3/2} \Big|_0^{64}$

$= \frac{1}{8}\,\frac{2}{3}\,64^{3/2} = \frac{2\;64(\cancel{8})}{\cancel{8}(3)} = \frac{2}{3}(64)$

$= \frac{128}{3}$

$= 42.6667$

years to live

on average

Illustrative Example 3

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)}$$

Suppose you are given that:

- $\mathring{e}_0 = 30$; and
- $S_0(x) = 1 - \dfrac{x}{\omega}$, for $0 \le x \le \omega$. limiting age

Evaluate $\mathring{e}_{15}$.

Solution to be discussed in lecture.

$$E(X) = \int_0^\infty S_0(x)\,dx$$

$$= \int_0^\omega \left(1 - \frac{x}{\omega}\right) dx$$

$$= \omega - \frac{1}{\cancel{\omega}} \cdot \frac{1}{2}(\omega^{\cancel{2}}) = \frac{\omega}{2}$$

$$= 30$$

$$\Rightarrow \omega = 60$$

$$\mathring{e}_{15} = E[T_{15}] = \int_0^\infty S_{15}(t)\,dt$$

$$\frac{S_0(15+t)}{S_0(15)} = \frac{45-t}{45}$$

$$= 1 - t/45$$

$$0 \le t \le 45$$

$$= \int_0^{45} (1 - t/45)\,dt = 45 - \frac{1}{\cancel{45}} \cdot \frac{1}{2}\, 45^{\cancel{2}}$$

$$= \frac{45}{2} = 22.5$$

$$S_0(x) = 1 - \frac{x}{\omega} \qquad f_0(x) = -\frac{d}{dx} S_0(x) = \frac{1}{\omega}, \quad 0 \le x \le \omega$$

uniform on $(0, \omega)$

$$E(X) = \frac{\omega}{2}$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{1 - \frac{x+t}{\omega}}{1 - \frac{x}{\omega}} = \frac{\frac{\omega - x - t}{\cancel{\omega}}}{\frac{\omega - x}{\cancel{\omega}}} = 1 - \frac{t}{\omega - x}, \quad 0 \le t \le \omega - x$$

$T_x \sim$ Uniform

distributed as

on $0 \le t \le \omega - x$

$$E[T_x] = \frac{\omega - x}{2}$$

De Moivre's law

Simplest Solution: $\frac{\omega}{2} = 30 \Rightarrow \omega = 60$

$$\mathring{e}_{15} = \frac{\omega - 15}{2} = \frac{60 - 15}{2} = 22.5$$

X is exponential with $\mu =$ constant

$$\mu e^{-x\mu} = f_0(x) = \quad , \quad x \geq 0$$

$$E(X) = \frac{1}{\mu} \qquad Var(X) = \frac{1}{\mu^2}$$

T_x is also exponential with $\mu =$ constant

$$f_x(t) = \mu e^{-\mu t} \rightarrow \text{independent of } x!$$

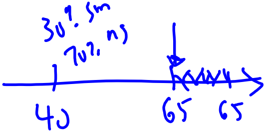
$$\Rightarrow \text{memoryless}$$

$$E(T_x) = \frac{1}{\mu}$$

$$S_0(x) = \int_x^\infty f_0(z)\,dz = e^{-\mu x}$$

$$S_x(t) = e^{-\mu t} = {}_tp_x$$

Illustrative Example 4



For a group of lives aged 40 consisting of 30% smokers (sm) and the rest, non-smokers (ns), you are given:

- For non-smokers, $\mu_x^{\text{ns}} = 0.05$, for $x \geq 40$ $\Rightarrow$ ${}_tp_{40}^{ns} = e^{-.05t}$
- For smokers, $\mu_x^{\text{sm}} = 0.10$, for $x \geq 40$ $\Rightarrow$ ${}_tp_{40}^{sm} = e^{-.10t}$

Calculate q_{65} for a life randomly selected from those who reach age 65.

Law of Total Probability -

$$q_{65} = q_{65}^{ns} \Pr[ns] + q_{65}^{sm} \Pr[sm]$$

$$(1-e^{-.05}) \qquad (1-e^{-.10})$$

$${}_1p_{65}^{ns} = e^{-.05}$$

$${}_1p_{65}^{sm} = e^{-.10}$$

$$\Pr[ns @ 65] = \frac{.70 \, {}_{25}p_{40}^{ns}}{.70 \, {}_{25}p_{40}^{ns} + .30 \, {}_{25}p_{40}^{sm}} = \frac{.70 \, e^{-.05(25)}}{.70 \, e^{-.05(25)} + .30 \, e^{-.10(25)}} = .8906403$$

$$\Pr[sm @ 65] = 1 - \Pr[ns @ 65] = \frac{.30 \, {}_{25}p_{40}^{sm}}{.70 \, {}_{25}p_{40}^{ns} + .30 \, {}_{25}p_{40}^{sm}} = .1093597$$

$$q_{65} = (1 - e^{-.05})(.8906403) + (1 - e^{-.10})(.1093597) = \underline{\underline{.05384399}}$$

Alternative:

$$q_{65} = 1 - p_{65} = 1 - \frac{S_0(66)/S_0(40)}{S_0(65)/S_0(40)}$$

$${}_tp_x = \frac{S_0(x+t)}{S_0(x)}$$

$$= 1 - \frac{S_{40}(26)}{S_{40}(25)} =$$

$$S_{40}(26) = \underset{= e^{-.05(26)}}{S_{40}^{ns}(26)} \underbrace{Pr[ns @ 40]}_{70\%} + \underset{e^{-.10(26)}}{S_{40}^{sm}(26)} \underbrace{Pr[sm @ 40]}_{30\%} = .2130543$$

$$S_{40}(25) = e^{-.05(25)}(70\%) + e^{-.10(25)}(30\%) = .2251789$$

$$= 1 - \frac{.2130543}{.2251789} = \underline{.0538439}$$

Temporary (partial) expectation of life

$\mathring{e}_0 \quad \mathring{e}_x$

We can also define temporary (or partial) expectation of life:

$$\mathsf{E}\big[\min(T_x, n)\big] = \mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$$

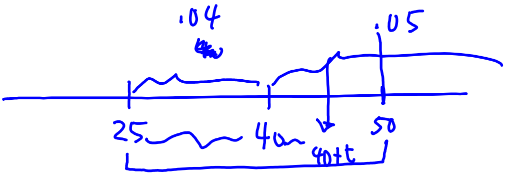
$\int_0^{\cancel{\infty}\, n} {}_tp_x dt \le n$

This can be interpreted as the average future lifetime of (x) within the next n years.

Suppose you are given:

$$\boxed{\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x \geq 40 \end{cases}}$$

Calculate $\mathring{e}_{25:\overline{25}|}$



$$\mathring{e}_{25:\overline{25}|} = \int_0^{25} {}_tp_{25}\,dt$$

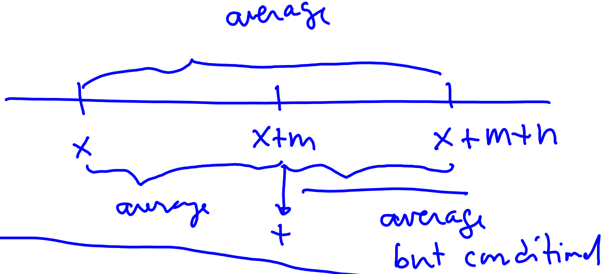
$${}_{15}p_{25} \cdot {}_tp_{40}$$

$$= \int_0^{15} {}_tp_{25}\,dt + \int_{15}^{25} {}_tp_{25}\,dt$$

$$= \int_0^{15} e^{-.04t}\,dt + {}_{15}p_{25} \int_0^{10} {}_tp_{40}\,dt$$

$$e^{-.04(15)} \int_0^{10} e^{-.05t}\,dt$$

$= \underline{15.59852 \text{ years}}$



$$\boxed{\overset{\circ}{e}_{x:\overline{m+n}|} = \overset{\circ}{e}_{x:\overline{m}|} + {}_{m}p_{x}\, \overset{\circ}{e}_{x+m:\overline{n}|}}$$

Generalized De Moivre's law

De Moivre = Uniform $S_0(x) = 1 - \frac{x}{\omega}$

The SDF of the so-called Generalized De Moivre's Law is expressed as

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \text{ for } 0 \le x \le \omega.$$

from birth

Derive the following for this special type of law of mortality:

1. force of mortality
2. survival function associated with T_x
3. expectation of future lifetime of x
4. can you find explicit expression for the variance of T_x?

$$T_0 = X \sim \underbrace{GDM}_{(\omega, \alpha)}, \text{ then } T_x \sim \underbrace{GDM}_{(\omega - x,\ \alpha)}$$

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} = \left(\frac{\omega - x}{\omega}\right)^{\alpha},$$

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} = \left(1 - \frac{t}{\omega - x}\right)^{\alpha}$$

$$\mu_x = \frac{-\frac{d}{dx} S_0(x)}{S_0(x)} = \frac{\cancel{-}\alpha \left(\frac{\omega - x}{\omega}\right)^{\cancel{\alpha}-1} \left(\cancel{-}\frac{1}{\omega}\right)}{\cancel{\left(\frac{\omega - x}{\omega}\right)^{\alpha}}} = \alpha \frac{\omega}{\omega - x} \cdot \frac{1}{\cancel{\omega}} = \frac{\alpha}{\omega - x},$$

$$\mu_{x+t} = \frac{\alpha}{\omega - x - t}$$

$$E(X) = \int_0^{\cancel{\infty}\ \omega} S_0(x)\,dx = \int_0^{\omega} \left(\frac{\omega - x}{\omega}\right)^{\alpha} dx = -\omega \int_1^0 u^{\alpha}\,du$$

$$u = \frac{\omega - x}{\omega}$$

$$du = -\frac{1}{\omega}\,dx$$

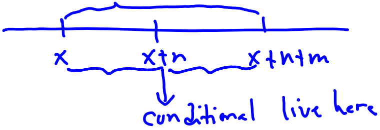
$$= -\omega \left.\frac{u^{\alpha+1}}{\alpha+1}\right|_1^0 = \frac{-\omega}{\alpha+1}(0-1)$$

$$= \frac{\omega}{\alpha+1}$$

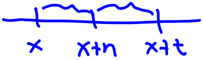
Similarly, one can deduce

$$E(T_x) = \frac{\omega - x}{\alpha+1} \quad \text{since } T_x \sim \text{GDM with } \omega - x, \alpha$$

$$\mathring{e}_{x:\overline{n+m}|} = \mathring{e}_{x:\overline{n}|} + {}_{n}p_{x}\, \mathring{e}_{x+n:\overline{m}|}$$



$$E\left[\min(T_x,\, n+m)\right]$$

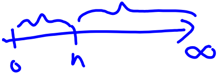


$$\int_0^{n+m} {}_{t}p_{x}\, dt = \underbrace{\int_0^{n} {}_{t}p_{x}\, dt} + \int_n^{n+m} {}_{t}p_{x}\, dt$$

$$= \mathring{e}_{x:\overline{n}|} + {}_{n}p_{x} \int_n^{n+m} {}_{t-n}p_{x+n}\, dt$$

$s = t-n$

$ds = dt$



$$= \mathring{e}_{x:\overline{n}|} + {}_{n}p_{x} \underbrace{\int_0^{m} {}_{s}p_{x+n}\, ds}_{\mathring{e}_{x+n:\overline{m}|}}$$

$m \to \infty$

$$\mathring{e}_{x} = \mathring{e}_{x:\overline{n}|} + {}_{n}p_{x}\, \mathring{e}_{x+n}$$

$$e_{x:\overline{n+m}|} = e_{x:\overline{n}|} + {}_np_x \, e_{x+n:\overline{m}|}$$

discrete analogue

$m \to \infty$

$$\underbrace{e_x = e_{x:\overline{n}|} + {}_np_x \, e_{x+n}}$$

approximate this integral

$$\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$

Illustrative example

check previous slides

- We will do **Example 2.6** in class.

Example 2.3

Gompertz μ increases exponentially with x!

$$c^s = e^{s \log c} = e^{s \ln c}$$

Let $\mu_x = Bc^x$, for $x > 0$, where B and c are constants such that $0 < B < 1$ and $c > 1$.

Derive an expression for $S_x(t)$.

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = e^{-\int_0^t \mu_{x+s}\, ds}$$

${}_tp_x$

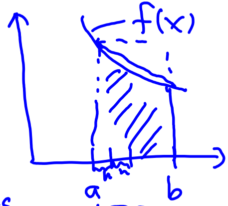
$$= e^{-\int_0^t Bc^{x+s}\, ds} = e^{-Bc^x \int_0^t e^{s(\log c)}\, ds}$$

$$= e^{-\frac{Bc^x}{\log c} c^s \Big|_0^t} = \underline{\underline{e^{-\frac{Bc^x}{\log c}(c^t - 1)}}}$$

$$E[T_x] = \mathring{e}_x = \int_0^\infty {}_tp_x\, dt$$

<u>Approximate integrals</u>

① $\int_a^b f(x)dx \approx (b-a)\,\frac{1}{2}[f(a)+f(b)]$



improve this integration by subdividing (a,b) into ~~X~~ sub-intervals of length h

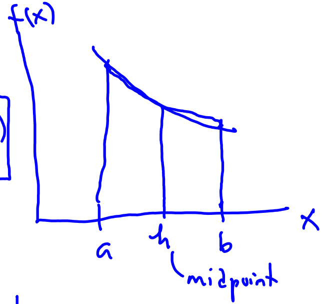
$\underbrace{b-a = n\cdot h}$

$$\int_a^b = \int_a^{a+h} + \int_{a+h}^{a+2h} + \cdots + \int_{a+(n-1)h}^{a+nh}$$

$$\int_0^5 = \int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 + \int_4^5$$

② Simpson's Rule

$$\int_a^b f(x)\,dx \approx \underbrace{\frac{h}{3}\left[f(a)+4f(a+h)+f(b)\right]}_{\text{length } 2h}$$



improve this: n intervals

$$\int_a^b = \int_a^{a+2h} + \int_{a+2h}^{a+4h} + \cdots + \int_{a+2(n-1)h}^{b}$$

0,5 $\quad h = 1/2$

$$\int_0^5 = \int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 + \int_4^5 =$$

$A = .002$

$B = 10^{-4.5}$

$C = 1.10$

$\mu_x = \underline{A} + \underline{Bc^x}$ Makeham's

$${}_tp_x = \underbrace{e^{-\int_0^t A\,ds}}_{e^{-At}} \; e^{-\frac{Bc^x}{\log c}(c^t - 1)}$$

$$\overset{\circ}{e}_{35:\overline{2}|} = \int_0^2 {}_tp_{35}\,dt = \underbrace{\int_0^2 e^{-.002t} \; e^{-\frac{10^{-4.5}(1.10)^{35}}{\log(1.10)}(1.1^t - 1)}\,dt}_{\int_0^1 + \int_1^2}$$

choose $h=1$ trapezoidal rule

t	${}_tp_{35}$
0	1
1	.9970719
2	.9940597

$$\approx \frac{1}{2}({}_0p_{35} + {}_1p_{35}) + \frac{1}{2}({}_1p_{35} + {}_2p_{35})$$

$$\approx \underline{\underline{1.994102}}$$

Apply Simpson's Rule $n=2$ $h=1/2$

$$\int_0^2 {}_tp_{35}\,dt = \underbrace{\int_0^1 {}_tp_{35}\,dt} + \int_1^2 {}_tp_{35}\,dt$$

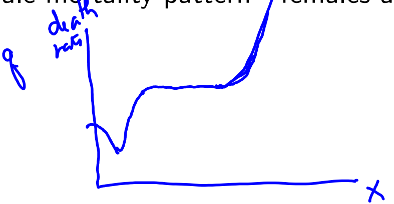
$$= \underbrace{\frac{1/2}{3}\left[\underset{1}{{}_0p_{35}} + {}_1p_{35} + 4\cdot \underset{.998546}{{}_{\frac{1}{2}}p_{35}}\right] + \frac{1/2}{3}\left[{}_1p_{35} + {}_2p_{35} + 4\cdot \underset{.9955768}{{}_{3/2}p_{35}}\right]}$$

$$= 1.994116$$

Exact Value $\int_0^2 {}_tp_{35}\,dt = \underline{1.994116}$ exactly - matches the Simpsons

Typical mortality pattern observed

- High (infant) mortality rate in the first year after birth.
- Average lifetime (nowadays) range between 70-80 - varies from country to country.
- Fewer lives/deaths observed after age 110 - supercentenarian is the term used to refer to someone who has reached age 110 or more.
- The highest recorded age at death (I believe) is 122.
- Different male/female mortality pattern - females are believed to live longer.



standard risk -

Substandard mortality

selection = underwriting

- A substandard risk is generally referred to someone classified by the insurance company as having a higher chance of dying because of:
  - some physical condition
  - family or personal medical history
  - risky occupation
  - dangerous habits or lifestyle (e.g. skydiving)
- Mortality functions are superscripted with s to denote substandard: q_x^s and μ_x^s.
- For example, substandard mortality may be obtained from a standard table using:
  1. adding a constant to force of mortality: $\mu_x^s = \mu_x + c$
  2. multiplying a fixed constant to probability: $q_x^s = \min(kq_x, 1)$
- The opposite of a substandard risk is preferred risk where someone is classified to have better chance of survival.

$$\mu_x^s = \mu_x + c$$

s = substandard

risky

$$ {}_tp_x^s = e^{-\int_0^t \mu_{x+z}^s \, dz} $$

$\mu_{x+z} + c$

$c > 0$

$$ = e^{-\int_0^t \mu_{x+z} dz} \, e^{-ct} $$

$$ < {}_tp_x $$

Substandard has worse mortality

$\Rightarrow$ worse survival

$q_x^s = k\, q_x \quad , k > 1$

$$ {}_tp_x^s = p_x^s \, p_{x+1}^s \, p_{x+2}^s \cdots p_{x+t-1}^s = \underbrace{(1-k\,q_x)}_{<1-q_x} \underbrace{(1-k\,q_{x+1})}_{1-q_{x+1}} \cdots\cdots $$

$$ < p_x \, p_{x+1} \cdots = {}_tp_x $$

Final remark - other contexts

human lifetime

time

- The notion of a lifetime or survival learned in this chapter can be applied in several other contexts:
  - engineering: lifetime of a machine, lifetime of a lightbulb
  - medical statistics: time-until-death from diagnosis of a disease, survival after surgery
  - finance: time-until-default of credit payment in a bond, time-until-bankruptcy of a company
  - space probe: probability radios installed in space continue to transmit
  - biology: lifetime of an organism
  - other actuarial context: disability, sickness/illness, retirement, unemployment

warranty

Other symbols and notations used

Expression	Other symbols used	
probability function	$P(\cdot)$ $\Pr(\cdot)$	$\Pr[\cdot]$
survival function of newborn	$S_X(x)$ $S(x)$ $s(x)$	$S_0(x)$
future lifetime of x	$T(x)$ T	T_x
curtate future lifetime of x	$K(x)$ K	K_x
survival function of x	$S_{T_x}(t)$ $S_T(t)$	$S_x(t)$
force of mortality of T_x	$\mu_{T_x}(t)$ $\mu_x(t)$	μ_{x+t}