Premium Calculation

Lecture: Weeks 12-14

Preliminaries

An insurance policy (life insurance or life annuity) is funded by contract premiums:

- once (single premium) made usually at time of policy issue, or
- a series of payments (usually contingent on survival of policyholder) with first payment made at policy issue
- to cover for the benefits, expenses associated with initiating/maintaining contract, profit margins, and deviations due to adverse experience.

Net premiums (or sometimes called benefit premiums)

- considers only the benefits provided
- nothing allocated to pay for expenses, profit or contingency margins

Gross premiums (or sometimes called expense-loaded premiums)

- covers the benefits and includes expenses, profits, and contingency margins

Chapter summary

- Contract premiums
  - net premiums
  - gross (expense-loaded) premiums
- Present value of future loss random variable
- Premium principles
  - the equivalence principle (or actuarial equivalence principle)
  - portfolio percentile premiums
- Return of premium policies
- Chapter 6 of Dickson, et al.

Net random future loss

- An insurance contract is an agreement between two parties:
  - the insurer agrees to pay for insurance benefits;
  - in exchange for insurance premiums to be paid by the insured.
- Denote by PVFB_0 the present value, at time of issue, of future benefits to be paid by the insurer.
- Denote by PVFP_0 the present value, at time of issue, of future premiums to be paid by the insured.
- The insurer's net random future loss is defined by

$$L_0^n = \mathsf{PVFB}_0 - \mathsf{PVFP}_0.$$

- Note: this is also called the present value of future loss random variable (in the book), and if no confusion, we may simply write this as L_0.

The principle of equivalence

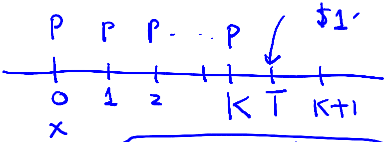
- The net premium, generically denoted by P, may be determined according to the principle of equivalence by setting

$$\mathsf{E}\left[L_0^n\right] = 0.$$

- The expected value of the insurer's net random future loss is zero.
- This is then equivalent to setting $\mathsf{E}\left[\mathsf{PVFB}_0\right] = \mathsf{E}\left[\mathsf{PVFP}_0\right]$. In other words, at issue, we have

$$\text{APV(Future Premiums)} = \text{APV(Future Benefits)}.$$

Whole life $\to$ discrete premiums $\to$ discrete   fully discrete



$$\boxed{L_0^n = L_0 = \underline{1 \cdot v^{K+1}} - P\ddot{a}_{\overline{K+1}|}} \qquad K = 0, 1, \ldots$$

$$\ddot{a}_{\overline{K+1}|} = \frac{1 - v^{K+1}}{d}$$

$$E[L_0] = E[v^{K+1}] - P\,E[\ddot{a}_{\overline{K+1}|}] = 0$$

$$P = \frac{E[v^{K+1}]}{E[\ddot{a}_{\overline{K+1}|}]} = \boxed{\frac{A_x}{\ddot{a}_x} = P_x}$$

$$L_0 = v^{K+1}\Big(\underbrace{1 + \frac{P}{d}}\Big) - \frac{P}{d} \Rightarrow Var[L_0] = \Big(1 + \frac{P}{d}\Big)^2\Big[{}^2A_x - (A_x)^2\Big]$$

$$1+\frac{P_x}{d} = 1+\frac{A_x/\ddot{a}_x}{d} = 1+\frac{A_x}{d\ddot{a}_x} = \frac{1}{1-A_x}$$

$A_x = 1-d\ddot{a}_x$

$d\ddot{a}_x = 1-A_x$

$$\boxed{Var[L_0] = \frac{{}^2A_x-(A_x)^2}{(1-A_x)^2}}$$

$E[L_0] = 0.$

$L_0 = PVFB_0 - PVFP_0$ $\quad A_{x:\overline{n}|} = 1-d\,\ddot{a}_{x:\overline{n}|}$

n-year endowment $\begin{cases} v^{K+1}, & K<n \\ v^n, & K\geq n\end{cases} - P\begin{cases}\ddot{a}_{\overline{K+1}|}, & K<n \\ \ddot{a}_{\overline{n}|}, & K\geq n\end{cases}$

- 1st prem immediately paid
- maturity if not started
- premium payment period < maturity

$$L_0^n = v^{\min(K+1,n)} - P\,\ddot{a}_{\overline{\min(K+1,n)}|}$$

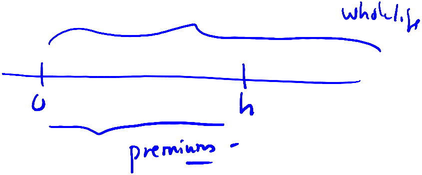
$$\frac{1-v^{\min(K+1,n)}}{d}$$

$E[L_0] = A_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|} = 0$

big P $\leftarrow P = \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = P_{x:\overline{n}|}$

$$Var(L_0) = \frac{{}^2A_{x:\overline{n}|}-(A_{x:\overline{n}|})^2}{(1-A_{x:\overline{n}|})^2}$$

h-pay whole life -



$$P = \frac{A_x}{\ddot{a}_{x:\overline{h|}}}$$

variance not explicit

n-year term

$$P = \frac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

n-year term, h-pay $\quad h \leq n$

$$P = \frac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{h|}}}$$

Remarks: ① $B \neq 1 \Rightarrow$ multiply $P * B \quad$ Var $\Rightarrow B^2$

② $APV(FB) = APV(FP)$

An illustration

Consider an n-year endowment policy which pays B dollars at the end of the year of death or at maturity, issued to a life with exact age x. Net premium of P is paid at the beginning of each year throughout the policy term.

- If we denote the curtate future lifetime of (x) by $K = K_x$, then the net random future loss can be expressed as
$$L_0^n = Bv^{\min(K+1,n)} - P\ddot{a}_{\overline{\min(K+1,n)|}}.$$
- The expected value of the net random future loss is
$$\begin{aligned} \mathsf{E}\left[L_0^n\right] &= B\mathsf{E}\left[v^{\min(K+1,n)}\right] - P\mathsf{E}\left[\ddot{a}_{\overline{\min(K+1,n)|}}\right] \\ &= BA_{x:\overline{n|}} - P\ddot{a}_{x:\overline{n|}}. \end{aligned}$$

An illustration - continued

- By the principle of equivalence, $\mathsf{E}\left[L_0^n\right] = 0$, we then have

$$P = B \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

- Rewriting the net random future loss as

$$L_0^n = \left(B + \frac{P}{d}\right) v^{\min(K+1,n)} - \frac{P}{d},$$

we can find expression for the variance:

$$\mathsf{Var}\left[L_0^n\right] = \left(B + \frac{P}{d}\right)^2 \left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right].$$

- One can also show that this simplifies to

$$\mathsf{Var}\left[L_0^n\right] = B^2 \frac{{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2}{\left(1 - A_{x:\overline{n}|}\right)^2}.$$

Some general principles

Note the following general principles when calculating premiums:

- For (discrete) premiums, the first premium is usually assumed to be made immediately at issue.
- Insurance benefit may have expiration or maturity:
  - in which case, it is implied that there are no premiums to be paid beyond expiration or maturity.
  - however, it is possible that premiums are to be paid for lesser period than expiration or maturity. In this case, it will be explicitly stated.

Fully discrete annual premiums - whole life insurance

Consider the case of a fully discrete whole life insurance where benefit of \$1 is paid at the end of the year of death with level annual premiums.

The net annual premium is denoted by P_x so that the net random future loss is

$$L_0 = v^{K+1} - P_x \, \ddot{a}_{\overline{K+1|}}, \quad \text{for } K = 0, 1, 2, \ldots$$

By the principle of equivalence, we have

$$P_x = \frac{\mathsf{E}\left[v^{K+1}\right]}{\mathsf{E}\left[\ddot{a}_{\overline{K+1|}}\right]} = \frac{A_x}{\ddot{a}_x}.$$

The variance of the net random future loss is

$$\mathsf{Var}[L_0] = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2} = \frac{{}^2A_x - (A_x)^2}{(1 - A_x)^2}.$$

Other expressions

You can express the net annual premiums:

- in terms of annuity functions

$$P_x = \frac{1 - d\ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - d$$

- in terms of insurance functions

$$P_x = \frac{A_x}{\big(1 - A_x\big)/d} = \frac{dA_x}{1 - A_x}$$

Whole life insurance with *h* premium payments

Consider the same situation where now this time there are only h premium payments.

- The net random future loss in this case can be expressed as

$$L_0 = v^{K+1} - P \times \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{for } K = 0, 1, \ldots, h-1 \\ \ddot{a}_{\overline{h}|}, & \text{for } K = h, h+1, \ldots \end{cases}$$

- Applying the principle of equivalence, we have

$$P = \frac{A_x}{\ddot{a}_{x:\overline{h}|}}.$$

Illustrative example 1

Consider a special endowment policy issued to (45). You are given:

- Benefit of \$10,000 is paid at the end of the year of death, if death occurs before 20 years.
- Benefit of \$20,000 is paid at the end of 20 years if the insured is then alive.
- Level annual premiums P are paid at the beginning of each year for 10 years and nothing thereafter.
- Mortality follows the Illustrative Life Table with $i = 6\%$.

Calculate P according to the equivalence principle.

$$APV(FB) = 10{,}000\, A^{1}_{45:\overline{20}|} + 20{,}000\ {}_{20}E_{45}$$

$$APV(FP) = P\, \ddot{a}_{45:\overline{10}|}$$

$$P = \frac{10{,}000\left[A^{\;1}_{45:\overline{20}|} + 2\ {}_{20}E_{45} \right]}{\ddot{a}_{45:\overline{10}|}}$$

p.12

$$A^{\;1}_{45:\overline{20}|} = .08846167 = A_{45} - {}_{20}E_{45}\, A_{65}$$

$${}_{20}E_{45} = 0.25634$$

$$\ddot{a}_{45:\overline{10}|} = \ddot{a}_{45} - {}_{10}E_{45}\, \ddot{a}_{55} = 7.648646$$

$$\underline{\underline{P = 758.9452}}$$

SOA-type question

benefit discrete e.o.y.
premium discrete b.o.y.

Two actuaries use the same mortality table to price a fully discrete two-year endowment insurance of 1,000 on (x). You are given:

- Kevin calculates non-level benefit premiums of 608 for the first year, and 350 for the second year.
- Kira calculates level annual benefit premiums of π.
- $d = 0.05$

Calculate π.

$APV(FP) = APVFB$

$608 + 350 v p_x = 1000 A_{x:\overline{2}|}$

$\pi \ddot{a}_{x:\overline{2}|} = 1000 A_{x:\overline{2}|} \Rightarrow 1 - d\ddot{a}_{x:\overline{2}|}$

$\Rightarrow \pi = 1000 \left[\frac{1}{\ddot{a}_{x:\overline{2}|}} - 0.05\right]$

$$\underbrace{608 + 350\, \underset{(\ddot{a}_{x:\overline{2}|}-1)}{v p_x} = 1000\,(1 - \underset{0.05}{d}\, \ddot{a}_{x:\overline{2}|})}$$

$$\underbrace{\ddot{a}_{x:\overline{2}|} = 1 + v p_x}$$

$$\ddot{a}_{x:\overline{2}|} = 742/400$$

$$\pi = 1000\left[\frac{1}{742/400} - .05\right] = \underline{489.08}$$

Illustrative example 2

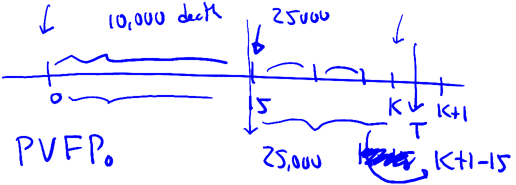
premiums benefits/annuity

0 15

An insurance company issues a 15-year deferred life annuity contract to (50). You are given:

- Level monthly premiums of P are paid during the deferred period.
- The annuity benefit of \$25,000 is to be paid at the beginning of each year the insured is alive, starting when he reaches the age of 65.
- Mortality follows the `Illustrative Life Table` with $i = 6\%$.
- Mortality between integral ages follow the Uniform Distribution of Death (UDD) assumption.

1. Write down an expression for the net future loss, at issue, random variable.
2. Calculate the amount of P.
3. If an additional benefit of \$10,000 is to be paid at the moment of death during the deferred period, how much will the increase in the monthly premium be?



① $L_0^n = PVFB_0 - PVFP_0$

$$= \begin{cases} \phi, & K<15 \\ 25000\, v^{15}\, \ddot{a}_{\overline{K+1-15}|}, & K\geq 15 \end{cases} - P \begin{cases} \ddot{a}^{(12)}_{\overline{K^{(12)}+\frac{1}{12}}|}, & K<15 \\ \ddot{a}_{\overline{15}|}, & K\geq 15 \end{cases}$$

② $\underbrace{APV(FP)} = \underbrace{APV(FB)}$

$$P\, \ddot{a}^{(12)}_{50:\overline{15}|} = 25000\; {}_{15}E_{50}\, \ddot{a}_{65} \Rightarrow P = \frac{25000\; {}_{15}E_{50}\, \ddot{a}_{65}}{\ddot{a}^{(12)}_{50:\overline{15}|}}$$

$$\ddot{a}^{(12)}_{50:\overline{15}|} = \ddot{a}^{(12)}_{50} - {}_{15}E_{50}\, \ddot{a}^{(12)}_{65}$$

$$\alpha(12)\, \ddot{a}_{50} - \beta(12) \qquad \alpha(12)\, \ddot{a}_{65} - \beta^{(12)}$$

$$P = 763.0536$$

$$\underbrace{APV(FP)} = APV(FB)$$

$$P\,\underbrace{\ddot{a}^{(12)}_{50:\overline{15}|}} = \underbrace{25000\,{}_{15}E_{50}\,\ddot{a}_{65}} + 10{,}000\,\underbrace{\bar{A}^{\,1}_{50:\overline{15}|}}$$

$$\frac{i}{\delta}A^{\,1}_{50:\overline{15}|}$$

UDD

$$\underbrace{\frac{i}{\delta}\left(A_{50} - {}_{15}E_{50}\,A_{65}\right)}_{0.09739964}$$

$i = 6\%$

$P = \ldots$

$$\text{Increase} = \frac{973.9964}{\ddot{a}^{(12)}_{50:\overline{15}|}} = \underline{\underline{8.552853}}$$

Different possible combinations

discrete

Premium payment	Benefit payment
annually	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death
m-thly of the year	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death
continuously	at the end of the year of death
	at the end of the $\frac{1}{m}$th year of death
	at the moment of death

discrete

fully continuous

Fully continuous premiums - whole life insurance

Consider a fully continuous level annual premiums for a unit whole life insurance payable immediately upon death of (x).

- The insurer's net random future loss is expressed as

$$L_0 = v^T - P\,\bar{a}_{\overline{T|}}.$$

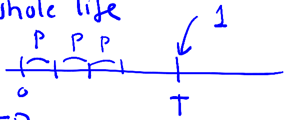
- By the principle of equivalence,

$$P = \frac{\bar{A}_x}{\bar{a}_x} = \frac{1}{\bar{a}_x} - \delta = \frac{\delta \bar{A}_x}{1 - \bar{A}_x}.$$

- The variance of the insurer's net random future loss can be expressed as

$$\begin{aligned} \mathsf{Var}[L_0] &= \left[1 + (P/\delta)\right]^2 \left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right] \\ &= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{(\delta \bar{a}_x)^2} = \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{(1 - \bar{A}_x)^2}. \end{aligned}$$

Fully continuous - whole life



$$L_0 = PVFB_0 - PVFP_0$$

issue time

$$= v^T - P\,\bar{a}_{\overline{T|}}$$

$$\bar{a}_{\overline{T|}} = \frac{1-v^T}{\delta}$$

$$= v^T\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}$$

$$\Rightarrow E[L_0] = \underbrace{\bar{A}_x - P\,\bar{a}_x = 0}$$

$$P = \frac{\bar{A}_x}{\bar{a}_x}$$

$$\boxed{Var[L_0] = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2}}$$

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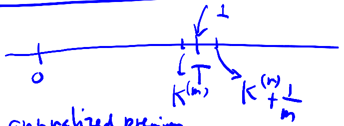
m^{thly} payment, continuous whole life



$$L_0 = v^T - P\,\ddot{a}^{(m)}_{\overline{K^{(m)}+\frac{1}{m}|}}$$

P = annualized premium

P/m = m^{thly} prem

A simple illustration

For a fully continuous whole life insurance of \$1, you are given:

- Mortality follows a constant force of $\mu = 0.04$.
- Interest is at a constant force $\delta = 0.08$.
- L_0 is the loss-at-issue random variable with the benefit premium calculated based on the equivalence principle.

Calculate the annual benefit premium and Var$[L_0]$.

Published SOA question #14

$\bar{A}_x = \frac{\mu}{\mu+\delta}$ $\bar{a}_x = \frac{1}{\mu+\delta}$

$P = \frac{\bar{A}_x}{\bar{a}_x} = \mu$

For a fully continuous whole life insurance of $1 on (x), you are given:

- The forces of mortality and interest are constant.
- ${}^2\bar{A}_x = 0.20$ $\rightarrow \frac{\mu}{\mu+2\delta} = 0.20 \Rightarrow \frac{.03}{.03+2\delta} = 0.20 \Rightarrow \frac{.03-.006}{.4} = \delta$ $.024/.4$ $.06$
- The benefit premium is 0.03. $= P = \mu$
- L_0 is the loss-at-issue random variable based on the benefit premium.

Calculate Var$[L_0]$. $\rightarrow$

$L_0 = v^T - P\bar{a}_{\overline{T|}} = \left(1+\frac{P}{\delta}\right)v^T - \frac{P}{\delta}$

$\mathrm{Var}[L_0] = \left(1+\frac{P}{\delta}\right)^2 \mathrm{var}(v^T)$

$0.20 = \left(1+\frac{.03}{.06}\right)^2 \left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$

$.20$ $\left(\frac{.03}{.09}\right)^2$

Endowment insurance

Consider an n-year endowment insurance with benefit of \$1:

- The net random future loss is

$$L = \begin{cases} v^T - P\,\bar{a}_{\overline{T}|}, & T \le n \\ v^n - P\,\bar{a}_{\overline{n}|}, & T > n \end{cases}$$

- Net annual premium formulas:

$$P = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} = \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta = \frac{\delta\,\bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}}$$

- The variance of the net random future loss:

$$\begin{aligned} \mathrm{Var}\big[L_0\big] &= \big[1+\big(P/\delta\big)\big]^2\left[{}^2\bar{A}_{x:\overline{n}|} - \big(\bar{A}_{x:\overline{n}|}\big)^2\right] \\ &= \frac{{}^2\bar{A}_{x:\overline{n}|} - \big(\bar{A}_{x:\overline{n}|}\big)^2}{(\delta\,\bar{a}_{x:\overline{n}|})^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - \big(\bar{A}_{x:\overline{n}|}\big)^2}{(1-\bar{A}_{x:\overline{n}|})^2} \end{aligned}$$

Illustrative example 3

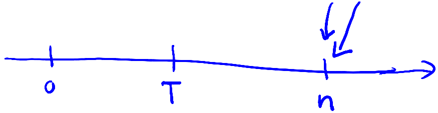
continuous benefit

continuous premium

For a fully continuous n-year endowment insurance of \$1 issued to (x), you are given:

- Z is the present value random variable of the benefit for this insurance.
- $\mathsf{E}[Z] = 0.5198$
- $\mathsf{Var}[Z] = 0.1068$
- Level annual premiums are paid on this insurance, determined according to the equivalence principle.

Calculate $\mathsf{Var}[L_0]$, where L_0 is the net random future loss at issue.



$$\begin{cases} v^T, & T<n \\ v^n, & T \geq n \end{cases} = v^{\min(T,n)}$$

$$\mathrm{Var}[L_0] = \left(1+\frac{P}{\delta}\right)^2 \underbrace{\mathrm{Var}[Z]}_{0.1068}$$

$$P = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} = \frac{\bar{A}_{x:\overline{n}|}}{\frac{1-\bar{A}_{x:\overline{n}|}}{\delta}}$$

$$\frac{P}{\delta} = \frac{\bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}} \quad E[Z]$$

$$L_0 = PVFB_0 - PVFP_0$$

$$= v^{\min(T,n)} - P\,\underbrace{\bar{a}_{\overline{\min(T,n)}|}}_{\frac{1-v^{\min(T,n)}}{\delta}}$$

$$L_0 = \underbrace{\left(1+\frac{P}{\delta}\right)}\,\underbrace{v^{\min(T,n)}}_{Z} - \underbrace{P/\delta}$$

$$= \left(1+\frac{.5198}{1-.5198}\right)^2 (.1068) = .4631556$$

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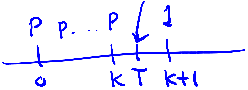
Loss at issue

$$\underset{\text{today}}{L_0} = \underbrace{PVFB_0}_{\substack{\text{insurance}\\ \text{or annuity}}} - \underbrace{PVFP_0}_{\substack{\text{annuity}\\ \text{or}\\ \text{single premium}}}$$

e.g. fully discrete whole life



$$L_0 = v^{K+1} - \underset{\substack{\text{or}\\ \Pi}}{P}\,\ddot{a}_{\overline{K+1}|}$$

Eq Principle: $E[L_0] = 0 \iff \boxed{APVFB_0 = APVFP_0}$.

$$P_x = \frac{A_x}{\ddot{a}_x} \iff v^{K+1} - P\left(\frac{1-v^{K+1}}{d}\right) = \underbrace{v^{K+1}\left(1+\frac{P}{d}\right)} - \frac{P}{d}$$

$$Var[L_0] = \left(1+\frac{P}{d}\right)^2 \underbrace{Var(v^{K+1})}_{\left[{}^2A_x - (A_x)^2\right]}$$

P = net premium

or benefit premium

e.g. fully continuous

$$L_0 = v^T - P\,\bar{a}_{\overline{T}|} \iff P = \frac{\bar{A}_x}{\bar{a}_x} \iff E[L_0] = 0$$

$$\Rightarrow Var[L_0] = \left(1+\frac{P}{\delta}\right)^2 \left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$$

$$L_0 = f(T \text{ or } K) \Rightarrow L_0 < 0 \iff T \text{ or } K \lessgtr t$$

e.g. fully continuous $\quad L_0 = v^T - P\,\bar{a}_{\overline{T|}}$

$$= v^T\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}$$

$T \sim$ constant force $\mu \implies$ Exponential with mean $1/\mu$

$f(t) = \mu e^{-\mu t}, \; t>0$

$P(T>t) = e^{-\mu t}$

$\bar{A}_x = \frac{\mu}{\mu+\delta}, \quad \bar{a}_x = \frac{1}{\mu+\delta}$

$\log v^T = T \log v$

$= -\delta T$

$P = \mu$

$$\Pr[L_0 < 0] = \Pr\left[v^T\left(1+\frac{P}{\delta}\right) < \frac{P}{\delta}\right]$$

$$= \underbrace{\Pr\left[T > -\frac{1}{\delta}\log\left(\frac{P/\delta}{1+P/\delta}\right)\right]}$$

$\underline{\text{Profit} = -L_0}$

$$= e^{-\frac{\mu}{\delta}\log\left(\frac{P/\delta}{1+P/\delta}\right)}$$

Illustrative example 4

For a fully discrete whole life insurance of 100 on (30), you are given:

- π denotes the annual premium and $L_0(\pi)$ denotes the net random future loss-at-issue random variable for this policy.
- Mortality follows the `Illustrative Life Table` with $i = 6\%$.

Calculate the smallest premium, π^*, such that the probability is less than 0.5 that the loss $L_0(\pi^*)$ is positive.

$L_0 = v^{K+1}\left(100 + \pi^*/d\right) - \pi^*/d$ $\quad d = 1 - v = i/1+i$

$\pi^* \to \underbrace{Pr[L_0 > 0]}_{Pr(K \gtrless\ )} < 0.50$

$$L_0 = v^{K+1}\left(100 + \frac{\pi^*}{d}\right) - \frac{\pi^*}{d} > 0$$

$$\Rightarrow v^{K+1} > \frac{\pi^*/d}{100 + \pi^*/d}$$

$\log v^{K+1}$

$(K+1) \underbrace{\log v}_{-\delta}$

$$\Rightarrow K < \underbrace{\frac{\log\left(\frac{\pi^*/d}{100+\pi^*/d}\right)}{-\delta} - 1}_{m}$$

$m-1 \quad m$

$$Pr[L_0 > 0] < 0.5 \Rightarrow Pr[\underbrace{K < m}] < 0.5$$

$$Pr[K \le k] = {}_{k+1}q_x \qquad \Rightarrow Pr[K \le m-1] < 0.5$$

$$\Rightarrow {}_m q_{30} < 0.5 \Rightarrow {}_m p_{30} \geq 0.5$$

$${}_m p_{30} = \frac{l_{30+m}}{l_{30}} \geq 0.5 \Rightarrow l_{30+m} \geq 0.5\, l_{30}$$

$$\underbrace{9501381}_{4750691}$$

$30 + m = 77 \Rightarrow m = 47$ years

$$l_{77} = 4828182$$

$$l_{78} = 4530360$$

Solve for π^*:

$$m = \frac{\log\left(\frac{\pi^*/d}{100 + \pi^*/d}\right)}{\cancel{\delta}} \cancel{t} = 4\cancel{7} - 4 \div \delta$$

Compare 0.64624 with $\pi = \frac{A_{30}}{a_{30}} \times 100$

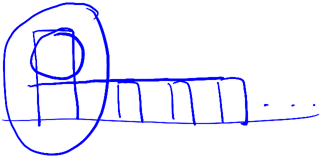
$$\boxed{e^{-45/\delta}\left(100 + \pi^*/d\right) = \pi^*/d}$$

$$\Rightarrow \quad \pi^* = d\left(\frac{100}{e^{+45/\delta} - 1}\right) = d\left[\frac{100}{e^{+488} - 1}\right] = 0.3677033$$

Types of Life Insurance contract expenses

- Investment-related expenses (e.g. analysis, cost of buying, selling, servicing)
- Insurance-related expenses:
  - acquisition (agents' commission, underwriting, preparing new records)
  - maintenance (premium collection, policyholder correspondence)
  - general (research, actuarial, accounting, taxes)
  - settlement (claim investigation, legal defense, disbursement)

First year vs. renewal expenses



- Most life insurance contracts incur large losses in the first year because of large first year expenses:
  - agents' commission
  - preparing new policies, contracts
  - records administration
- These large losses are hopefully recovered in later years.
- How then do these first year expenses spread over the policy life?
- Anything not first year expense is called renewal expense (used for maintaining and continuing the policy).

Gross premium calculations

- Treat expenses as if they are a part of benefits. The gross random future loss at issue is defined by

$$L_0^g = \mathsf{PVFB}_0 + \mathsf{PVFE}_0 - \mathsf{PVFP}_0,$$

where PVFE_0 is the present value random variable associated with future expenses incurred by the insurer.

- The gross premium, generically denoted by G, may be determined according to the principle of equivalence by setting

$$\mathsf{E}\left[L_0^g\right] = 0.$$

- This is equivalent to setting $\mathsf{E}\left[\mathsf{PVFB}_0\right] + \mathsf{E}\left[\mathsf{PVFE}_0\right] = \mathsf{E}\left[\mathsf{PVFP}_0\right]$. In other words, at issue, we have

$$\boxed{\mathsf{APV}(\mathsf{FP}_0) = \mathsf{APV}(\mathsf{FB}_0) + \mathsf{APV}(\mathsf{FE}_0).}$$

Illustration of gross premium calculation

A 1,000 fully discrete whole life policy issued to (45) with level annual premiums is priced with the following expense assumptions:

	% of Premium	Per 1,000	Per Policy
First year	40%	1.0	5.0
Renewal years	10%	0.5	2.5

In addition, assume that mortality follows the *Illustrative Life Table* with interest rate $i = 6\%$.

Calculate the expense-loaded annual premium. = gross annual premium

$$APVFP_0 = APVFB_0 + APVFE_0$$

$$G\,\ddot{a}_{45}$$

$$1000\,A_{45}$$

$$.30G + .10G\,\ddot{a}_{45}$$ % of Premium

$$.50 + .50\,\ddot{a}_{45}$$ per 1000

$$2.50 + 2.50\,\ddot{a}_{45}$$ per policy

$$G\left(\cancel{.90\,\ddot{a}_{45} - .30}\right) = \frac{1000\,A_{45}' + 3 + 3\,\ddot{a}_{45}'}{.90\,\ddot{a}_{45} - 0.30}$$

$$G = 19.88/\text{year}$$

$A_{45} = .20120$

$\ddot{a}_{45} = 14.1121$

ILT

Published SOA question #239 - modified

For a 20-year endowment insurance of 25,000 on (x) with benefit payable at the moment of death, you are given:

- Expenses are incurred at the beginning of the year:

	Percent of Premium	Per 1,000 of Insurance	Per Policy
First year	25%	2.0	15.0
Renewal	5%	0.5	3.0

- $\bar{A}_{x:\overline{20}|} = 0.4058$
- $\ddot{a}_{x:\overline{20}|} = 12.522$
- $i = 0.05$
- Level annual premiums are determined using the equivalence principle.

Calculate the level annual expense-loaded premium.

$$APVFP_0 = G\,\ddot{a}_{x:\overline{20|}}$$

25,000

$$\left.\begin{matrix}2\\ .5\end{matrix}\right) 1.5 \qquad \begin{matrix}15\\ 3\end{matrix}$$

$$APVFB_0 = 25{,}000\,\bar{A}_{x:\overline{20|}}$$

$+$

$$APVFE_0 = \underline{.20G} + .05G\,\ddot{a}_{x:\overline{20|}}$$

$$+ \frac{1.50}{1000}\cdot 25\cancel{000} + .5(25)\,\ddot{a}_{x:\overline{20|}}$$

$$+ 12 + 3\,\ddot{a}_{x:\overline{20|}}$$

$$\cancel{G\left(.95\,\ddot{a}_{x:\overline{20|}} - .20\right)} = \frac{25000\,\bar{A}_{x:\overline{20|}} + 49.5 + 15.5\,\ddot{a}_{x:\overline{20|}}}{.95\,\ddot{a}_{x:\overline{20|}} - .20}$$

$$G = 888.225$$

Portfolio percentile premium principle

Suppose insurer issues a portfolio of N "identical" and "independent" policies where the PV of loss-at-issue for the i-th policy is $L_{0,1}$.

The total portfolio (aggregate) future loss is then defined by

$$L_{\mathsf{agg}} = L_{0,1} + L_{0,2} + \cdots + L_{0,N} = \sum_{i=1}^{N} L_{0,i}$$

Its expected value is therefore

$$\mathsf{E}[L_{\mathsf{agg}}] = \sum_{i=1}^{N} \mathsf{E}\left[L_{0,i}\right]$$

and, by "independence", the variance is

$$\mathsf{Var}[L_{\mathsf{agg}}] = \sum_{i=1}^{N} \mathsf{Var}\left[L_{0,i}\right].$$

Portfolio percentile premium principle

The portfolio percentile premium principle sets the premium P so that there is a probability, say α with $0 < \alpha < 1$, of a positive gain from the portfolio.

In other words, we set P so that

$$\Pr[L_{\text{agg}} < 0] = \alpha.$$

Note that loss could include expenses.

Consider Example 6.11 (1st edition) or Example 6.12 (2nd edition)

Illustrative example 5

An insurer sells 100 fully discrete whole life insurance policies of $1, each of the same age 45. You are given:

- All policies have independent future lifetimes.
- Mortality follows the `Standard Select Survival Model` with $i = 5\%$.

Using the Normal approximation:

1. Calculate the annual contract premium according to the portfolio percentile premium principle with $\alpha = 0.95$.
2. Suppose the annual contract premium is set at 0.010 per policy. Determine the smallest number of policies to be sold so that the insurer has at least a 95% probability of a gain from this portfolio of policies.

Illustration of return of premium

To illustrate the concept of return of premium policies, consider Question #22 from Fall 2012 SOA MLC Exam:

You are given the following information about a special fully discrete 2-payment, 2-year term insurance on (80):

- Mortality follows the `Illustrative Life Table`.
- $i = 0.0175$
- The death benefit is 1000 plus a return of all premiums paid without interest.
- Level premiums are calculated using the equivalence principle.

Calculate the benefit premium for this special insurance.

For practice: try calculating the benefit premium if the return of all premiums paid comes with an interest of say 0.01.

Other terminologies and notations used

Expression	Other terms/symbols used
net random future loss	loss-at-issue
L_0	${}_0L$
net premium	benefit premium
equivalence principle	actuarial equivalence principle
generic premium (P)	π