MATH 3630

Actuarial Mathematics I

Sample Test 1

Time Allowed: 1 hour

Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: EMIL Student ID: SUGGESTED SOLUTIONS

- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.

Question No. 1:

The force of mortality for a newborn is given by

$$\mu_x = \frac{1}{460 - cx}, \quad \text{for } 0 \le x < \omega,$$

for some positive constant c.

You are also given: $\mathring{e}_{35} = 64$

Calculate ${}_{10}q_{35}$ and interpret this value.

One may recognize μ_x as that of a Generalized de Moivre's

$\mu_x = \dfrac{\alpha}{\omega - x}$, by rewriting $\mu_x = \dfrac{1}{460 - cx} = \dfrac{1/c}{\frac{460}{c} - x}$ so that

$\alpha = 1/c$ and $\omega = 460/c$.

Similarly, for Generalized de Moivre's, $\mathring{e}_x = \dfrac{\omega - x}{\alpha + 1}$ so that

$$\mathring{e}_{35} = \frac{\frac{460}{c} - 35}{\frac{1}{c} + 1} = \frac{460 - 35c}{1 + c} = 64.$$

Solving for c, we get $460 - 35c = 64 + 64c$

$$\underbrace{460 - 64}_{396} = (64 + 35)c = 99c$$

$$\Rightarrow c = \frac{396}{99} = 4$$

*CORRECTION noted

$\therefore \alpha = 1/4$ and $\omega = 115$

Thx to JB

$${}_{10}q_{35} = 1 - {}_{10}p_{35} = 1 - \left(1 - \frac{10}{115 - 35}\right)^{1/4} = 1 - \left(\frac{7}{8}\right)^{1/4} = 1 - 0.9671682 = 0.0328318 \checkmark$$

This gives the probability that a life age 35 now will die within the next ten years.

Remark: If someone forgot property of GDM, one can always work this out from first principles. Use the formulas

$$\mathring{e}_x = \int_0^{\omega - x} {}_tp_x \, dt \quad \text{and} \quad {}_tp_x = e^{-\int_0^t \mu_{x+s} \, ds}.$$

Question No. 2:

You are given:

- The probability that (50) dies before reaching age 65 is 0.117.
- The probability that (50) dies between the ages of 65 and 75 is 0.185.
- $\ell_{75} = 65310$

Calculate ℓ_{50}.

$${}_{15}q_{50} = 0.117 \Rightarrow {}_{15}p_{50} = 1 - 0.117 = 0.883$$

$${}_{15|10}q_{50} = 0.185 \Rightarrow {}_{15|10}q_{50} = {}_{15}p_{50} - {}_{25}p_{50} = 0.185$$

$$\Rightarrow {}_{25}p_{50} = {}_{15}p_{50} - 0.185$$

$$= 0.883 - 0.185$$

$$= .698$$

$$\frac{l_{75}}{l_{50}} = \frac{65310}{l_{50}} = 0.698 \Rightarrow l_{50} = \frac{65310}{.698} = \underline{\underline{93567}}$$

Question No. 3:

Assume the constant force of mortality assumption holds between integral ages. You are given:

$S_0(65) = 0.730$

$S_0(66) = 0.725$

$S_0(67) = 0.720$

* correction noted

Calculate the probability that a person who is currently age 65 years and 4 months will survive at least one more year.

$$p_{65\frac{4}{12}} = p_{65\frac{1}{3}} = \frac{\ell_{66\frac{1}{3}}}{\ell_{65\frac{1}{3}}} = \frac{\ell_{66}^{2/3}\,\ell_{67}^{1/3}}{\ell_{65}^{2/3}\,\ell_{66}^{1/3}}$$ by constant force assumption

$$= \frac{\ell_{66}^{2/3}\,\ell_{67}^{1/3}/\ell_0}{\ell_{66}^{1/3}\cdot\ell_{65}^{2/3}/\ell_0} = \frac{S_0(66)^{2/3}\;S_0(67)^{1/3}}{S_0^{2/3}(65)\;S_0(66)^{1/3}}$$

$$\cancel{= \left(\frac{S_0(66)}{S_0(65)}\right)^{1/3} = \left(\frac{.725}{.730}\right)^{1/3} = 0.99771}$$

$$= \frac{S_0(66)^{1/3}\,S_0(67)^{1/3}}{S_0(65)^{2/3}} = \frac{(.725)^{1/3}(.720)^{1/3}}{(.730)^{2/3}} = 0.993135$$

EV corrected Oct 5, 2011

Question No. 4:

You are given:

- $\ell_x = 100(\omega - x)^{1/2}$, for $0 \le x \le \omega$
- $\mathring{e}_{40} = 40$
- T_{40} is the future lifetime random variable for a person currently age 40.

Calculate $\mathrm{Var}[T_{40}]$.

$$S_{40}(t) = \frac{\ell_{40+t}}{\ell_{40}} = \frac{100\,(\omega-40-t)^{1/2}}{100\,(\omega-40)^{1/2}} = \left(1-\frac{t}{\omega-40}\right)^{1/2}$$

So T_{40} has the form of a Generalized de Moivre's with $\alpha = 1/2$

$$E[T_{40}] = \mathring{e}_{40} = \frac{\omega-40}{\alpha+1} = \frac{2}{3}(\omega-40) = 40 \Rightarrow \omega = 100$$

$$S_{40}(t) = \left(1-\frac{t}{60}\right)^{1/2} \Rightarrow f_{40}(t) = -\frac{d}{dt}S_{40}(t) = \frac{1}{2}\frac{1}{\sqrt{60}}(60-t)^{-1/2}$$

$$\therefore E[T_{40}^2] = \int_0^{60} t^2 \frac{1}{2\sqrt{60}}(60-t)^{-1/2}\,dt$$, Change variable of integration $s = 60-t$, $ds = -dt$

$$= -\int_{60}^{0} (-s+60)^2 \frac{1}{2\sqrt{60}} s^{-1/2}\,ds = \frac{1}{2}\frac{1}{\sqrt{60}}\int_0^{60} \underbrace{s^{-1/2}(s^2-120s+3600)}_{s^{3/2}+120s^{1/2}+3600s^{-1/2}}\,ds$$

$$= \frac{1}{\cancel{2}\,\cancel{\sqrt{60}}}\left(\frac{\cancel{2}}{5}(60)^{5/2} - \frac{\cancel{2}(120)}{3}(60)^{3/2} + \cancel{2}(3600)\cancel{(60)^{1/2}}\right) = 1920$$

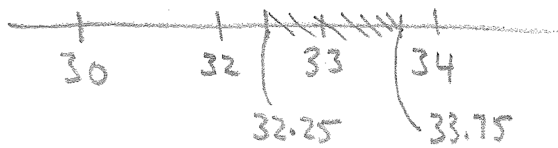
$$\mathrm{Var}[T_{40}] = E[T_{40}^2] - \mathring{e}_{40}^{\,2} = 1920 - 40^2 = \underline{\underline{320}}$$

Question No. 5:

Assume mortality follows the *Illustrative Life Table.*

Suppose that Uniform Distribution of Death (UDD) assumption holds between integral ages.

Calculate ${}_{2.25|1.5}q_{30}$ and interpret this probability.



$$ {}_{2.25|1.5}q_{30} = {}_{2.25}p_{30} - {}_{3.75}p_{30} $$

$$ = {}_{2}p_{30}\,{}_{.25}p_{32} - {}_{3}p_{30}\,{}_{.75}p_{33} $$

$$ = \frac{\cancel{l_{32}}}{l_{30}} \cdot \frac{l_{32.25}}{\cancel{l_{32}}} - \frac{\cancel{l_{33}}}{l_{30}} \frac{l_{33.75}}{\cancel{l_{33}}} = \frac{(l_{32.25} - l_{33.75})}{l_{30}} $$

$$ = \left[(.75\,l_{32} + \cancel{.25\,l_{33}}) - (\cancel{.25\,l_{33}} + .75\,l_{34})\right] / l_{30} $$

$$ = .75(9471591 - 9438571)/9501381 $$

$$ = \underline{\underline{0.002606463}} $$

This gives the probability that (30) will die between ages 32.25 and 33.75.

Question No. 6:

The mortality for Joshua, age 30, follows de Moivre's law with $\omega = 105$.

In the coming year, Joshua intends to join an expedition for mountain climbing. If he does, his assumed mortality will be adjusted so that for the coming year only, he will have a force of mortality increased by 50%.

Calculate the decrease in his 5-year temporary complete life expectation if he pursues this expedition.

* correction noted

Without mountain climbing, $\mathring{e}_{30:\overline{5}|} = \int_0^5 {}_tp_{30}\,dt = \int_0^5 \left(1-\frac{t}{75}\right)dt$

$$= 5 - \frac{1}{75}\left(\frac{1}{2}\right)(5^2) = 4.83333$$

With mountain climbing,

$$\mathring{e}^{*}_{30:\overline{5}|} = \mathring{e}^{*}_{30:\overline{1}|} + p^{*}_{30}\,\mathring{e}_{31:\overline{4}|} = \int_0^1 \left(1-\frac{t}{75}\right)^{1.5} dt + \left(\frac{74}{75}\right)^{1.5}\int_0^4 \left(1-\frac{t}{74}\right)dt$$

$$= -75\left(\frac{2}{5}\right)\left(1-\frac{t}{75}\right)^{5/2}\Big|_0^1 + \left(\frac{74}{75}\right)^{1.5}\left(4 - \frac{1}{74}\left(\frac{1}{2}\right)(4^2)\right)$$

$$= 4.804336$$

decrease $= 4.83333 - 4.804336$

$= 0.028994$ years

EV corrected Oct 5, 2011

~~At first glance, looks like a big drop in life expectancy, but this could be explained by the fact that survival without mountain climbing in the first year is $p_{30} = \frac{1}{75} = .0133$, while that with mountain climbing is $p^{*}_{30} = \left(\frac{1}{75}\right)^{1.5} = .0015$. A big fall!!!~~

ignore this comment because I calculated p_{30} incorrectly. It should be $p_{30} = \frac{74}{75} = \underline{0.9867}$ vs $p^{*}_{30} = \left(\frac{74}{75}\right)^{1.5} = \underline{0.9801}$

This drop in survival in the first year leads to a reduction of average life in the next 5 years by about 0.03 years! $\Longleftarrow$

Question No. 7:

For a certain population, you are given:

- The mortality for males follows a Gompertz law with $B = 9.0 \times 10^{-5}$ and $c = 1.09$.
- The mortality for females follows a Gompertz law with $B = 4.5 \times 10^{-5}$ and $c = 1.10$.
- Immediately after 30 years from birth, 40% are males and 60% are females.

Calculate the probability that a randomly chosen 30-year-old from this population will die within the next 10 years.

$${}_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - \left(.40\ {}_{10}p_{30}^{male} + .60\ {}_{10}p_{30}^{female} \right)$$

For Gompertz law, ${}_tp_x = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{e^{-B/\log c\,(c^{x+t}-1)}}{e^{-B/\log c\,(c^x-1)}}$

$$= e^{-\frac{B}{\log c}(c^t-1)c^x}$$

$${}_{10}p_{30}^{male} = \exp\left[-\frac{9\times10^{-5}}{\log(1.09)} (1.09^{10}-1)(1.09)^{30} \right] = 0.981232$$

$${}_{10}p_{30}^{female} = \exp\left[-\frac{4.5\times10^{-5}}{\log(1.10)} (1.10^{10}-1)(1.10)^{30} \right] = 0.9869556$$

$${}_{10}q_{30} = 1 - (.4(.981232) + .6(.9869556))$$

$$= \underline{0.0153385}$$

Question No. 8:

For a given life age 40, it is estimated that an impact of a medical breakthrough will be an increase of 3.2 years in $\mathring{e}_{40}$.

Prior to the medical breakthrough, mortality followed a Generalized de Moivre's law with $\alpha = 0.5$ and limiting age $\omega = 105$.

After the medical breakthrough, a Generalized de Moivre's law for mortality still applies with the same parameter α, but with a different limiting age.

Calculate the new limiting age as a result of the medical breakthrough.

Before medical breakthrough, $\mathring{e}_{40} = \dfrac{\omega - 40}{\alpha + 1} = \dfrac{105-40}{1.5} = 43.33333$

After medical breakthrough,

$$\mathring{e}^{*}_{40} = \frac{\omega^* - 40}{\alpha + 1} = \frac{\omega^* - 40}{1.5} = 43.33333 + 3.2$$

$$\therefore \omega^* = 40 + 1.5(46.53333)$$

$$= 109.8 \approx \underline{\underline{110}}$$

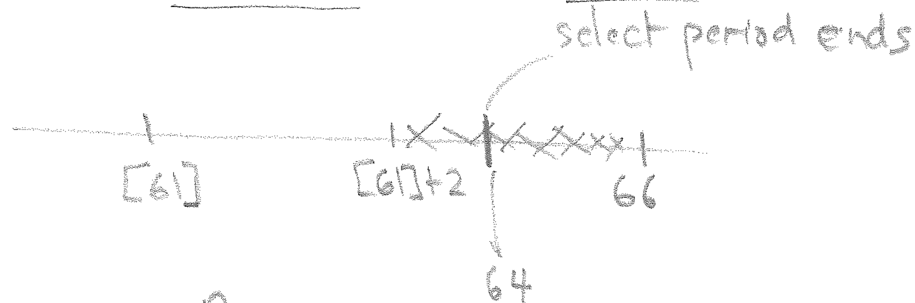
an increase of 5 years in the limiting age!

Question No. 9:

Suppose you are given the following select-and-ultimate mortality table:

$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	x+3
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

Calculate the probability that a life with select age 61 will survive for two years but die the following three years.



$$ {}_{2|3}q_{[61]} = {}_2p_{[61]} - {}_5p_{[61]} $$

$$ = (1-q_{[61]})(1-q_{[61]+1}) - (1-q_{[61]})(1-q_{[61]+1})(1-q_{[61]+2}) \times (1-q_{64})(1-q_{65}) $$

$$ = (.90)(.88)\left[1 - .86(.84)(.83)\right] $$

$$ = \underline{\underline{0.3171231}} $$

Question No. 10:

The force of mortality for a substandard life (x) is expressed as

$$\mu^*_{x+t} = \mu_{x+t} + k,$$

for some positive constant k, where μ_{x+t} is the force of mortality of a standard life (x).

You are given:

- The probability that a standard life (x) survives another year is 0.80.
- The probability that a substandard life (x) will die within the following year is 0.34.

Calculate k.

$p_x = 0.80 \quad q^*_x = 0.34 \Rightarrow p^*_x = 0.66$

Since $\mu^*_{x+t} = \mu_{x+t} + k \Rightarrow p^*_x = p_x e^{-k}$

$\therefore \quad (.66) = (.80) e^{-k}$

$k = -\log(66/80)$

$= \underline{\underline{0.1923719}}$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK