MATH 3630
Actuarial Mathematics I
Sample Finals
Fall 2011
Time Allowed: 2 hours
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: Emil **Student ID:** Suggested Solutions

- **There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.**
- **Please provide details of your workings in the appropriate spaces provided; partial points will be granted.**
- **Please write legibly.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Good luck.**

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	÷ 120	

Question No. 1:

For a fully discrete whole life insurance of \$1,000 issued to (x), you are given:

- $\mu_{x+k} = 0.01$, for all $k \geq 0$; and
- $\delta = 3\%$

Calculate the level annual benefit premium for this policy.

$$ {}_kp_x = e^{-\int_0^k \mu_{x+s}\,ds} = e^{-.01k} $$

$$ \ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x = \sum_{k=0}^{\infty} e^{-.03k} e^{-.01k} = \sum_{k=0}^{\infty} e^{-.04k} = \frac{1}{1-e^{-.04}} $$

The level annual benefit premium is

$$ 1000P_x = 1000\frac{\overbrace{A_x}^{1-d\ddot{a}_x}}{\ddot{a}_x} = 1000\left[\frac{1}{\ddot{a}_x} - d\right] $$

$$ d = 1 - v = 1 - e^{-.03} $$

$$ = 1000\left[\cancel{1} - e^{-.04} - \cancel{1} + e^{-.03}\right] $$

$$ = 1000\left(-e^{-.04} + e^{-.03}\right) $$

$$ = \underline{\underline{9.656094}} $$

Question No. 2:

You are given:

- Mortality follows Makeham's law with $A = 0.00003$, $B = 0.0001$ and $c = 1.12$.
- $\ddot{a}_{50}^{(12)} = 7.7140$ based on Woolhouse's approximation with two terms.
- $\ddot{a}_{50}^{(12)} = 7.7064$ based on Woolhouse's approximation with three terms.

Calculate $\ddot{a}_{50}^{(12)}$ based on the Uniform Distribution of Death (UDD) assumption.

$$\underbrace{\ddot{a}_{50}^{(12)}\,W2 - \ddot{a}_{50}^{(12)}\,W3}_{\substack{7.7140 - 7.7064 \\ .0076}} = \underbrace{\frac{12^2-1}{12(12)^2}}_{\frac{143}{1728}}(\delta + \mu_{50})$$

$$\mu_{50} \to A + Bc^{50} = .00003 + (.0001)(1.12)^{50} = .02893022$$

Solving for δ, we get

$$\delta = \frac{1728}{143}(.0076) - .02893022 = .06290754$$

Substituting back to $\underbrace{\ddot{a}_{50}^{(12)}\,W2}_{7.7140} = \ddot{a}_{50} - \frac{11}{24} \Rightarrow \ddot{a}_{50} = 8.172333$

$i = e^{\delta} - 1 = .06492837$

$i^{(12)} = 12\left[(1+i)^{1/12} - 1\right] = .06307272$

$d^{(12)} = 12\left[1 - (1+i)^{-1/12}\right] = .06274294$

$d = 1 - 1/(1+i) = .06096971$

so that

$$\ddot{a}_{50}^{(12)}\,\text{UDD} = \frac{i}{i^{(12)}}\frac{d}{d^{(12)}}\ddot{a}_{50} - \frac{(i - i^{(12)})}{i^{(12)}d^{(12)}}$$

Substituting the values, we get

$$\ddot{a}_{50}^{(12)}\,\text{UDD} = \underline{\underline{7.706099}}$$

Question No. 3:

Barbie Dahl and Bill Board are two actuaries who use the same mortality table to price a fully discrete two-year endowment insurance of 100 on (40). You are given:

- Barbie calculates level annual benefit premiums of 47.42.
- Bill calculates non-level benefit premiums of 50 for the first year, and π for the second year.
- Interest rate i is 8%.
- Both actuaries calculate benefit premiums based on the equivalence principle.

Calculate π.

$d = .08/1.08$

Barbie: $$47.42 = 100\,\frac{\overbrace{A_{40:\overline{2|}}}^{1-d\ddot{a}_{40:\overline{2|}}}}{\ddot{a}_{40:\overline{2|}}} \Rightarrow \ddot{a}_{40:\overline{2|}} = \frac{1}{.4742 + .08/1.08} = 1.823905$$

Bill:

$$\begin{array}{ccc} 50 & \pi & \\ \hline 40 & 41 & 42 \end{array}$$

$$50 + \pi \underbrace{v p_{40}}_{(\ddot{a}_{40:\overline{2|}}-1)} = 100 \underbrace{A_{40:\overline{2|}}}_{(1-d\ddot{a}_{40:\overline{2|}})}$$

$$\pi = \frac{100\left(1 - \frac{.08}{1.08}(1.823905)\right) - 50}{.823905}$$

$$= \underline{\underline{44.28857}}$$

Question No. 4:

For a fully discrete whole life insurance of \$1 issued to (x), you are given:

- K is the curtate future lifetime of (x).
- $\mathrm{E}\left[\ddot{a}_{\overline{K+1}|}\right] = 13.6$ $\qquad \mathrm{Var}\left[\dfrac{1-v^{K+1}}{d}\right] = \dfrac{1}{d^2}\mathrm{Var}\left[v^{K+1}\right]$
- $\mathrm{Var}\left[\ddot{a}_{\overline{K+1}|}\right] = 23.0$
- L_0 denotes the loss-at-issue random variable where premium is determined according to the equivalence principle.

Calculate $\mathrm{Var}[L_0]$.

$$\ddot{a}_x = \mathrm{E}\left[\ddot{a}_{\overline{K+1}|}\right] = 13.6$$

$$L_0 = v^{K+1} - P_x\,\ddot{a}_{\overline{K+1}|} = \left(1+\frac{P_x}{d}\right)v^{K+1} - \frac{P_x}{d}$$

$$\mathrm{Var}[L_0] = \left(1+\frac{P_x}{d}\right)^2 \mathrm{Var}\left[v^{K+1}\right] \qquad \left(1+\frac{A_x}{d\ddot{a}_x}\right) = \frac{d\ddot{a}_x + A_x}{d\ddot{a}_x} = \frac{1}{d\ddot{a}_x}$$

$$= \left(\frac{1}{d\ddot{a}_x}\right)^2 \mathrm{Var}\left[v^{K+1}\right]$$

$$= \frac{1}{d^2}\frac{1}{(\ddot{a}_x)^2}\mathrm{Var}\left[v^{K+1}\right]$$

$$= \frac{1}{\cancel{d^2}}\frac{1}{(13.6)^2}\cdot \cancel{d^2}(23.0) = \frac{23}{(13.6)^2} = \underline{\underline{0.1243512}}$$

Question No. 5:

You are given:

k	$\ddot{a}_{\overline{k+1}\rvert}$	${}_{k\rvert}q_x$
0	1.000	0.25
1	1.962	0.20
2	2.886	0.15
3	3.775	0.10

Calculate $\ddot{a}_{x:\overline{3}\rvert}$.

The PV of a 3-year temporary life annuity-due is given by

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}\rvert}, & K<3 \\ \ddot{a}_{\overline{3}\rvert}, & K\geq 3 \end{cases}$$

$$E[Y] = \ddot{a}_{x:\overline{3}\rvert} = 1(.25) + 1.962(.20) + 2.886(1-.25-.20)$$

$$= \underline{\underline{2.2297}}$$

* Corrected Dec 15, 2011
Thanks to KJ

Question No. 6:

Mortality follows Makeham's law with parameters

$A = 0.0027$

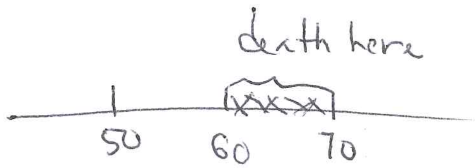
$B = 0.000018$

$c = 1.04$

Calculate ${}_{10|10}q_{50}$ and interpret this probability.

For Makeham's, ${}_tp_{50} = e^{-\int_0^t \mu_{50+s}\,ds} = e^{-\int_0^t (A + Bc^{50}c^s)\,ds}$

$= e^{-At}\, e^{-\frac{Bc^{50}}{\log c}(c^t - 1)}$



${}_{10|10}q_{50} = {}_{10}p_{50} - {}_{20}p_{50}$

$= e^{-10A}\, e^{-\frac{Bc^{50}}{\log c}(c^{10} - 1)} - e^{-20A}\, e^{-\frac{Bc^{50}}{\log c}(c^{20} - 1)}$

plug values A, B, c, we get

$= .02807926$

This gives the probability that a 50-year-old will survive for 10 years and die the following 10 years, or die between ages 60 and 70.

Question No. 7:

For a whole life insurance policy of \$100 issued to (x) where the benefit is payable at the end of the year of death, you are given:

- The single benefit premium for this policy is \$52.78.
- The level annual benefit premium for this same policy is \$4.80.
- The level annual benefit premium for the first 10 years, then reduced by 50% after 10 years and thereafter, for this same policy is \$5.70.

Calculate $A_{x:\overline{10}|}$.

$$100 A_x = 52.78 \Rightarrow A_x = .5278$$

$$4.80 = \frac{100 A_x}{\ddot{a}_x} = \frac{52.78}{\ddot{a}_x} \Rightarrow \ddot{a}_x = \frac{52.78}{4.80} = 10.99583$$

$$\underbrace{\quad x \quad \longrightarrow \quad x+10 \quad}_{5.70} \quad \underbrace{\quad x+10 \longrightarrow \quad}_{\frac{1}{2}(5.70)}$$

$$5.70 = \frac{100 \overset{.5278}{A_x}}{\underset{10.99583}{\ddot{a}_x} - \frac{1}{2}\, {}_{10}E_x\, \ddot{a}_{x+10}} \Rightarrow \underset{{}_{10|}\ddot{a}_x}{{}_{10}E_x\, \ddot{a}_{x+10}} = \underbrace{\left(10.99583 - \frac{52.78}{5.70}\right)\times 2}_{3.472368}$$

$$\ddot{a}_{x:\overline{10}|} = \ddot{a}_x - {}_{10|}\ddot{a}_x = 10.99583 - 3.472368 = 7.523465$$

$$\boxed{A_x = 1 - d\ddot{a}_x \qquad \frac{1-A_x}{\ddot{a}_x} = d}$$

$$A_{x:\overline{10}|} = 1 - d\,\ddot{a}_{x:\overline{10}|}$$

$$= 1 - \frac{(1-A_x)}{\ddot{a}_x}\ddot{a}_{x:\overline{10}|}$$

$$= 1 - \frac{(1-.5278)}{10.99583}(7.523465) = \underline{\underline{0.6769158}}$$

Question No. 8:

Get-a-Life Insurance Company decided to sell fully discrete whole life insurance of $1,000 to individuals age 40, something it has not done before. The actuaries of the company have determined that:

- For each policy, the annual benefit premium will be $7.50.
- Mortality follows the `Standard Select Survival Model` with $i = 5\%$.
- All the policyholders have independent future lifetimes.

Using Normal approximation, calculate the smallest number of policies the company must sell so that the probability of a positive loss at issue, on the aggregate, does not exceed 0.05.

Let n = number of policies to sell so that aggregate loss at issue is $L = L_1 + L_2 + \dots + L_n$

where $L_i = 1000 v^{K+1} - 7.50\, \ddot{a}_{\overline{K+1}|}$ $\quad \left(\ddot{a}_{\overline{K+1}|} = \dfrac{1-v^{K+1}}{d}\right)$

$$= \left(1000 + \frac{7.50}{d}\right) v^{K+1} - \frac{7.50}{d}, \quad i = 1, 2, \dots, n$$

and $K = K_{[40]}$ = curtate future lifetime of $[40]$

$$E[L_i] = \left(1000 + \frac{7.50}{d}\right) A_{[40]} - \frac{7.50}{d} = -17.47337$$

$d = .05/1.05$, $A_{[40]} = 1 - d\,\ddot{a}_{[40]}$, $d = \dfrac{.05}{1.05}$, $\ddot{a}_{[40]} = 18.45956$

$$Var[L_i] = \left(1000 + \frac{7.50}{d}\right)^2 \left({}^2A_{[40]} - \left(A_{[40]}\right)^2\right) = 11717.21$$

$E[L] = n E[L_i]$ and $Var[L] = n\, Var[L_i]$

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$$\Pr[L>0] \leq .05 \Rightarrow \Pr\left[Z > \frac{17.47337\,n}{\sqrt{11717.21\,n}}\right] \leq .05$$

$$\Rightarrow \quad \frac{17.47337\,\cancel{n}^{\sqrt{n}}}{\sqrt{11717.21\cancel{n}}} \geq 1.645$$

$$\sqrt{n} \geq \frac{1.645\sqrt{11717.21}}{17.47337} = 10.19064$$

$$n \geq 103.8491$$

Company must therefore sell at least 104 policies!

Question No. 9:

Suppose $i = 4\%$ and you are given the following extract from a select-and-ultimate mortality table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
40	75000	74511	73874	73123	43
41	74171	73668	73017	72245	44
42	73318	72801	72132	71341	45
43	72444	71910	71223	70409	46
44	71545	70994	70286	69448	47
45	70616	70046	69315	68451	48

Calculate $1000P_{[40]:\overline{3}|}$.

$$\ddot{a}_{[40]:\overline{3}|} = 1 + v\,p_{[40]} + v^2\,p_{[40]}\,p_{[40]+1}$$

$$= 1 + v\frac{\ell_{[40]+1}}{\ell_{[40]}} + v^2\frac{\ell_{[40]+2}}{\ell_{[40]}} = \underbrace{1 + (1.04)^{-1}\frac{74511}{75000} + (1.04)^{-2}\frac{73874}{75000}}_{2.865945}$$

$$A_{[40]:\overline{3}|} = 1 - d\,\ddot{a}_{[40]:\overline{3}|}$$

$$1000P_{[40]:\overline{3}|} = 1000\left[\frac{1}{\ddot{a}_{[40]:\overline{3}|}} - d\right] \qquad d = \frac{.04}{1.04}$$

$$= \underline{\underline{310.4635}}$$

Question No. 10:

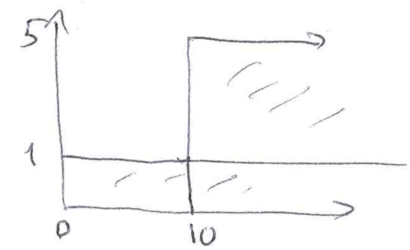
Brady, currently age 40, purchases a special whole life insurance that will pay him at the end of the year of his death the following benefits:

- \$1 if death occurs within the first 10 years, and
- \$5 if death occurs thereafter.

You are given:

- Level annual benefit premiums of P are to be paid at the beginning of each year for 10 years.
- Mortality follows the `Illustrative Life Table`.
- $i = 6\%$

Calculate P.



$$APV(\text{premiums}) = APV(\text{benefits})$$

$$P\,\underbrace{\ddot{a}_{40:\overline{10|}}}_{\ddot{a}_{40} - {}_{10}E_{40}\,\ddot{a}_{50}} = A_{40} + 4\,{}_{10}E_{40}\,A_{50}$$

$$P = \frac{\overset{.16132}{A_{40}} + 4\,\overset{.53667}{{}_{10}E_{40}}\,\overset{.24905}{A_{50}}}{\underset{14.8166}{\ddot{a}_{40}} - {}_{10}E_{40}\,\underset{13.2668}{\ddot{a}_{50}}}$$

$$= \frac{0.6959507}{7.696706} = \underline{\underline{0.09042188}}$$

Question No. 11:

You are given:

- ${}_{20}P_{35} = 0.0137$
- $P_{35:\overline{20|}} = 0.0329$
- $A_{55} = 0.3895$

Calculate $P^{\,1}_{35:\overline{20|}}$.

$$\Rightarrow \quad {}_{20}P_{35} = \frac{A_{35}}{\ddot{a}_{35:\overline{20|}}} = .0137$$

$$\frac{A_{35} = A^{1}_{35:\overline{20|}} + {}_{20}E_{35}\, A_{55}}{\ddot{a}_{35:\overline{20|}}}$$

$$.0137 = P^{1}_{35:\overline{20|}} + \frac{{}_{20}E_{35}}{\ddot{a}_{35:\overline{20|}}}(.3895)$$

$$\frac{A_{35:\overline{20|}} = A^{1}_{35:\overline{20|}} + {}_{20}E_{35}}{\ddot{a}_{35:\overline{20|}}}$$

$$\underbrace{P_{35:\overline{20|}}}_{.0329} - P^{1}_{35:\overline{20|}} = \frac{{}_{20}E_{35}}{\ddot{a}_{35:\overline{20|}}}$$

$$.0137 = P^{1}_{35:\overline{20|}} + (.0329 - P^{1}_{35:\overline{20|}})(.3895)$$

Solving for $P^{1}_{35:\overline{20|}}$, we get

$$(1 - .3895)P^{1}_{35:\overline{20|}} = .0137 - .0329(.3895)$$

$$P^{1}_{35:\overline{20|}} = \frac{.00088545}{.6105} = \underline{\underline{.00145}}$$

Question No. 12:

For a special whole life insurance issued to (40), you are given:

- Benefit is payable at the moment of death.
- $b_t = 100\,e^{0.04t}$, for $t \geq 0$
- Mortality follows de Moivre's law with $\omega = 110$.
- $\delta = 5\%$

Calculate the actuarial present value of the death benefits for this policy.

$T_{40} \sim$ Uniform on $(0, 70)$ $\quad f_{40}(t) = \frac{1}{70}, \quad 0 \leq t \leq 70$

$$\text{APV(benefits)} = E[b_T v^T]$$

$$= \int_0^{70} 100\, e^{.04t}\, e^{-.05t}\, \underbrace{f_{40}(t)}_{1/70}\, dt$$

$$= \frac{100}{70} \int_0^{70} e^{-.01t}\, dt$$

$$= \frac{100}{70} \frac{1}{.01} \left(1 - e^{-.01(70)}\right)$$

$$= \underline{\underline{71.91639}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK