MATH 3630
Actuarial Mathematics I
Final Examination - sec 001
Monday, 10 December 2012
Time Allowed: 2 hours (6:00 - 8:00 pm)
Room: MSB 411
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** SUGGESTED SOLUTIONS

- There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.
- Good luck.
- Have a Happy and Healthy Christmas and New Year!

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	÷ 120	

Question No. 1:

You are given:

$${}_tp_x = e^{-\frac{B}{\log c}c^x(c^t-1)}$$

- Mortality follows Gompertz' law, $\mu_x = Bc^x$, with $B = 0.0001$ and $c = 1.10$.
- $\delta = 5\%$
- $\ddot{a}_{50} = 12.09$
- $\ddot{a}_{65} = 7.55$

Evaluate $\ddot{a}^{(4)}_{50:\overline{15|}}$ based on Woolhouse's approximation with three terms and interpret this value.

$$\underbrace{\ddot{a}^{(4)}_{50} \approx \underset{12.09}{\ddot{a}_{50}} - \frac{4-1}{8} - \frac{16-1}{12(16)}(.05 + \mu_{50})}_{= 11.71018} \qquad .0001(1.1)^{50} = .01173909$$

$$.0001(1.1)^{65} = .04903707$$

$$\underbrace{\ddot{a}^{(4)}_{65} \approx \underset{7.55}{\ddot{a}_{65}} - \frac{3}{8} - \frac{15}{12(16)}(.05 + \mu_{65})}_{= 7.167263}$$

$${}_{15}E_{50} = v^{15}\,{}_{15}p_{50} = e^{-.05(15)}\,e^{-\frac{B}{\log c}c^{50}(c^{15}-1)} = .3193931$$

$$\ddot{a}^{(4)}_{50:\overline{15|}} = \ddot{a}^{(4)}_{50} - {}_{15}E_{50}\,\ddot{a}^{(4)}_{65}$$

$$\approx \underbrace{11.71018 - (.3193931)(7.167263)}$$

$$= \underline{\underline{9.421002}}$$

This gives the actuarial present value of a 15-year temporary life annuity issued to (50) of \$1 per year payable at the beginning of each quarter.

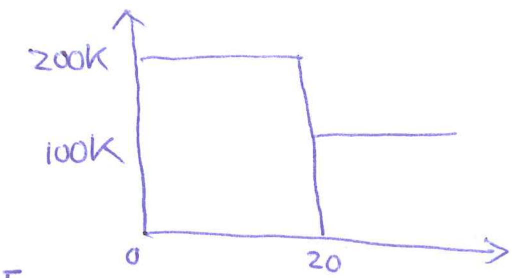
Question No. 2:

For a special whole life insurance policy issued to (45), you are given:

- Death benefits are payable at the end of the year of death while level premiums are payable once a year.
- The amount of benefit is \$200,000 if death occurs within the first 20 years and reduces to \$100,000 for death thereafter.
- Mortality follows the `Illustrative Life Table`.
- $i = 6\%$

Calculate the annual benefit premium based on the actuarial equivalence principle.

Let P be the annual benefit premium



$$P\ddot{a}_{45} = 200{,}000\, A_{45} - 100{,}000\; {}_{20}E_{45}\, A_{65}$$

$$P = 100{,}000 \left[\frac{2A_{45} - {}_{20}E_{45} A_{65}}{\ddot{a}_{45}} \right]$$

$A_{45} = .20120$

$A_{65} = .43980$

${}_{20}E_{45} = .25634$

$\ddot{a}_{45} = 14.1121$

$$= 100{,}000 \left[\frac{2(.20120) - (.25634)(.43980)}{14.1121} \right]$$

$$= \underline{\underline{2{,}052.577}}$$

Question No. 3:

For a fully discrete whole life insurance policy of \$1,000 on (x), you are given:

- The annual benefit premium is
$$P^* = P + 2,$$
where P is the annual benefit premium determined according to the equivalence principle.
- $i = 0.10$
- $A_x = 0.5$
- ${}^2A_x = 0.3$

Calculate the variance of the loss-at-issue.

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{A_x}{\frac{1-A_x}{d}} = \frac{\frac{.10}{1.10}(.5)}{1-.5} = .0909091$$

$$P = 1000(.0909091) = 90.90909$$

$$P^* = 92.90909$$

$$L_0 = 1000v^{K+1} - P^* \ddot{a}_{\overline{K+1|}}$$

$$= \left(1000 + \frac{P^*}{d}\right)v^{K+1} - \frac{P^*}{d}$$

$$Var[L_0] = \left(1000 + \frac{92.90909}{.10/1.10}\right)^2 \underbrace{Var[v^{K+1}]}_{{}^2A_x - (A_x)^2}$$

$$= \left(1000 + \frac{92.90909(1.10)}{.10}\right)^2 \left(.3 - (.5)^2\right)$$

$$= \underline{\underline{204,424.20}}$$

Question No. 4: constant μ, constant δ $\quad \bar{a}_x = \frac{1}{\mu+\delta}$

For a whole life annuity on (x) with benefits continuously paid at the rate of \$12,000 per year, you are given:

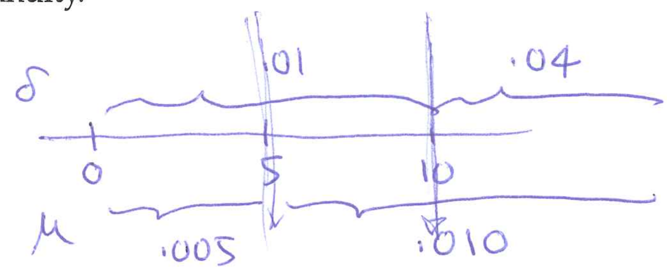
$$\delta_t = \begin{cases} 0.01, & 0 < t \le 10 \\ 0.04, & t > 10 \end{cases}$$

$\bar{a}_{x:\overline{n}|} = \frac{1}{\mu+\delta}\left(1-e^{-(\mu+\delta)n}\right)$

and

$$\mu_{x+t} = \begin{cases} 0.005, & 0 < t \le 5 \\ 0.010, & t > 5 \end{cases}$$

Calculate the actuarial present value for this annuity.



$$APV(\text{annuity}) = \left(\frac{1}{.015}\left(1-e^{-.015(5)}\right) + e^{-.015(5)} \cdot \frac{1}{.02}\left(1-e^{-.02(5)}\right) + e^{-.015(5)} \cdot e^{-.02(5)} \frac{1}{.05}\right) * 12{,}000$$

$$= 26.02056 * 12000$$

$$= \underline{\underline{312{,}246.8}}$$

Question No. 5: Let P = Semi-annual premium

For a special two-year term life insurance policy issued to (40), you are given:

- Benefit of \$10,000 is payable at the moment of death.
- Premiums are payable semi-annual.
- The following is an extract from the mortality table:

x	40	41	42
ℓ_x	100	98	95

$\Rightarrow \ell_{x+t} = \ell_x^{1-t}\, \ell_{x+1}^{t}$, $0<t<1$

- Mortality follows a constant force assumption over each year of age.
- $i^{(2)} = 0.06$ [Note: The corresponding force of interest is $\delta = 0.05912$.]
- Premiums are based on the actuarial equivalence principle.

P P P P

40 40.5 41 41.5 42

Calculate the amount of the semi-annual premium for this policy.

From the given table, we note that

x	ℓ_x
40	100
40.5	99
41	98
41.5	96.5
42	95

$$\ell_{40.5} = 100^{.5}\, 98^{.5} = 98.99495$$

$$\ell_{41.5} = 98^{.5}\, 95^{.5} = 96.48834$$

and

$$p_{40} = e^{-\mu_{40}} \Rightarrow \mu_{40} = -\log\frac{\ell_{41}}{\ell_{40}} = .02020271$$

$$p_{41} = e^{-\mu_{41}} \Rightarrow \mu_{41} = -\log\frac{\ell_{42}}{\ell_{41}} = .03109059$$

Applying equivalence principle, we have $APV(FP_0) = APV(FB_0)$

where
$$APV(FP_0) = P\left[1 + \frac{v^{1/2}\ell_{40.5} + v\,\ell_{41} + v^{3/2}\ell_{41.5}}{\ell_{40}}\right]$$

$$= P\left[1 + \frac{(1.03)^{-1}(99) + (1.03)^{-2}(98) + (1.03)^{-3}(96.5)}{100}\right]$$

$$= P(3.767865)$$

THIS PAGE FOR EXTRA SPACE TO SOLVE QUESTION 5

$\mu_{40}+\delta = .07932271$

$\mu_{41}+\delta = .09021059$

and

$$APV(FB_0) = 10{,}000 \int_0^2 v^t \, {}_tp_{40} \, \mu_{40+t} \, dt$$

$$= 10{,}000 \left[\int_0^1 e^{-\delta t} e^{-\mu_{40} t} \mu_{40} \, dt + e^{-(\mu_{40}+\delta)} \int_0^1 e^{-\delta t} e^{-\mu_{41} t} \mu_{41} \, dt \right]$$

$$= 10{,}000 \left[\underbrace{\frac{.02020271}{.07932271}\left(1 - e^{-.07932271}\right)}_{.01942221} + \underbrace{\frac{.03109059}{.09021059} e^{-.07932271}\left(1 - e^{-.09021059}\right)}_{.02746236} \right]$$

$$= 468.8457$$

Solving for P, we get

$$P = \frac{468.8457}{3.767865} = \underline{\underline{124.4327}}$$

Question No. 6:

For a fully discrete whole life insurance of $1 issued to (x), you are given:

- L_0 is the net loss-at-issue random variable with the premium determined according to the actuarial equivalence principle.
- L_0^* is the net loss-at-issue random variable with the premium determined such that $\mathrm{E}[L_0^*] = -0.5$.
- $\mathrm{Var}[L_0] = 0.75$.

Calculate $\mathrm{Var}[L_0^*]$.

$P = \frac{A_x}{\ddot{a}_x}$ when $E[L_0]=0 \Rightarrow L_0 = \left(1+\frac{P}{d}\right)v^{K+1} - P/d$

$P^* = \frac{A_x + .5}{\ddot{a}_x}$ when $E[L_0^*] = -.5 \Rightarrow L_0^* = \left(1+\frac{P^*}{d}\right)v^{K+1} - P^*/d$

$$\frac{\mathrm{Var}[L_0^*]}{\mathrm{Var}[L_0]} = \frac{(1+P^*/d)^2 \cancel{\mathrm{Var}[v^{K+1}]}}{(1+P/d)^2 \cancel{\mathrm{Var}[v^{K+1}]}} = \left(\frac{d+P^*}{d+P}\right)^2$$

$$= \left(\frac{\frac{d\ddot{a}_x + A_x + .5}{\cancel{\ddot{a}_x}}}{\frac{d\ddot{a}_x + A_x}{\cancel{\ddot{a}_x}}}\right)^2$$

$$= (1.5)^2$$

$\Rightarrow \mathrm{Var}[L_0^*] = (1.5)^2\,\mathrm{Var}[L_0] = 1.5^2(.75)$

$= \underline{\underline{1.6875}}$

Question No. 7:

Get-a-Life Insurance Company issues a special 10-year endowment insurance policy to a person now age 45. You are given:

- Premiums are payable once a year, where the first year premium is P and the subsequent premiums are each year equal to half of the first year.
- If death occurs within the first 10 years, the amount of benefit is \$1,000,000 plus the difference between the first and second year premiums.
- An endowment equal to the first year premium is paid at the end of 10 years.
- Mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$.

Calculate P.

$$APV(FP_0) = \tfrac{1}{2}P + \tfrac{1}{2}P\,\ddot{a}_{45:\overline{10|}}$$

$$APV(FB_0) = \left(1000000 + \tfrac{1}{2}P\right)A^{1}_{45:\overline{10|}} + P\,{}_{10}E_{45}$$

where

$$\ddot{a}_{45:\overline{10|}} = \underset{17.8162}{\ddot{a}_{45}} - \underset{.60655}{{}_{10}E_{45}}\,\underset{16.0599}{\ddot{a}_{55}} = 8.075068$$

$$A^{1}_{45:\overline{10|}} = \underset{.15161}{A_{45}} - \underset{.60655}{{}_{10}E_{45}}\,\underset{.23524}{A_{55}} = .008925178$$

Solving for P, we get

$$\tfrac{1}{2}P\left(1 + \ddot{a}_{45:\overline{10|}} - A^{1}_{45:\overline{10|}} - 2\,{}_{10}E_{45}\right) = 1000000\,A^{1}_{45:\overline{10|}}$$

$$P = \frac{2000000(.008925178)}{1 + 8.075068 - .008925178 - 2(.60655)}$$

$$= \frac{17850.36}{7.853042} = \underline{\underline{2,273.05}}$$

Question No. 8:

For a cohort of individuals all age x consisting of non-smokers (ns) and smokers (sm), you are given:

- Mortality is based on the following:

k	q^{ns}_{x+k}	q^{sm}_{x+k}
0	0.01	0.08
1	0.03	0.12

- $i = 5\%$
- $A^{\,1}_{x:\overline{2}|} = 0.0616$ for a randomly chosen individual from this cohort

Determine the proportion of non-smokers and smokers in this cohort at age x.

$${}^{ns}A^{\,1}_{x:\overline{2}|} = v q^{ns}_x + v^2 p^{ns}_x q^{ns}_{x+1} = \frac{1}{1.05}(.01) + \frac{1}{1.05^2}(.99)(.03) = .03646259$$

$${}^{sm}A^{\,1}_{x:\overline{2}|} = v q^{sm}_x + v^2 p^{sm}_x q^{sm}_{x+1} = \frac{1}{1.05}(.08) + \frac{1}{1.05^2}(.92)(.12) = .1763265$$

Let ns% be the proportion of non-smokers at age x

$$A^{\,1}_{x:\overline{2}|} = ns\% \; {}^{ns}A^{\,1}_{x:\overline{2}|} + (1 - ns\%)\; {}^{sm}A^{\,1}_{x:\overline{2}|}$$

$$.0616 = ns\% (.03646259 - .1763265) + .1763265$$

$$\Rightarrow ns\% = \frac{.0616 - .1763265}{.03646259 - .1763265} = 82\%$$ are non-smokers

so that the rest 18% are smokers!!

Question No. 9:

For a special 3-year temporary life annuity on (65), you are given:

- The annuity payments are \$20, \$30, and \$50, respectively, payable at the beginning of each year while (65) is alive. No further payments made after 3 years.
- Mortality is based on the following extract from a life table:

deaths 100 200 300

x	65	66	67	68
ℓ_x	9500	9400	9200	8900

- $i = 4\%$
- Y is the present value random variable for this annuity.

$v = \frac{1}{1.04}$

Calculate Var[Y].

K	$\Pr[K_{65}=k]$	y (p.v. of benefits)		$y * \Pr[K_{65}=k]$	$y^2 * \Pr[K_{65}=k]$
0	100/9500	20	20	.2105263	4.210526
1	200/9500	$20+30v$	48.84615	1.0283401	50.230458
≥ 2	9200/9500	$20+30v+50v^2$	95.07396	92.0716288	8753.614765
		Σ		93.3105	8808.056

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2$$

$$= 8808.056 - (93.3105)^2$$

$$= \underline{\underline{101.2072}}$$

Question No. 10:

Christian is currently age 45 who purchases a special deferred whole life annuity-due policy which will pay him the following benefits:

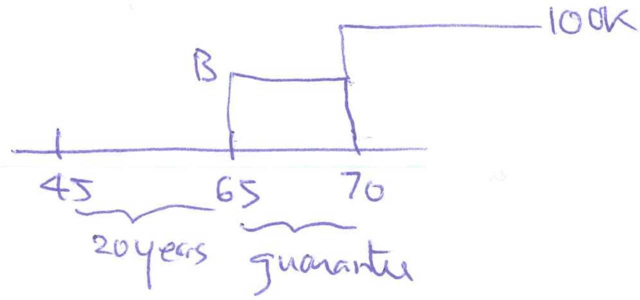
- guaranteed annual payments of B for 5 years, with the first payment to start when he reaches age 65; and
- annual payments of \$100,000, if alive, thereafter.

You are given:

- Mortality follows the `Illustrative Life Table.`
- $i = 6\%$
- The actuarial present value of Christian's life annuity benefits is \$225,000.

Calculate B.

$$\ddot{a}_{\overline{5}|} = \frac{1-v^5}{d} = \frac{1-(1/1.06)^5}{.06/1.06} = 4.465106$$



$$\underbrace{\text{APV(Christian's annuity)}}_{225,000} = B\,\ddot{a}_{\overline{5}|}\;{}_{20}E_{45} + 100,000\;{}_{25}E_{45}\,\ddot{a}_{70}$$

$${}_{20}E_{45} = .25634$$

$${}_{25}E_{45} = {}_{20}E_{45}\;{}_{5}E_{65} = \overbrace{(.25634)(.65623)}^{.168218}$$

Solving for B, we get

$$B = \frac{225,000 - 100,000\;{}_{25}E_{45}\,\overbrace{\ddot{a}_{70}}^{8.5693}}{\ddot{a}_{\overline{5}|}\;{}_{20}E_{45}}$$

$$= \frac{225,000 - 100,000(.168218)(8.5693)}{(4.465106)(.25634)} = \frac{80848.95}{1.144585} = 70,636.02$$

Question No. 11:

For a fully discrete whole life insurance of $1 on (50), you are given:

- Mortality follows the `Illustrative Life Table.`
- $i = 6\%$
- The annual benefit premium is equal to 0.02.

Calculate the probability of a positive loss-at-issue.

Loss-at-issue $= L_0 = \left(1 + \frac{.02}{d}\right) v^{K+1} - \frac{.02}{d}$

$$\Pr[L_0 > 0] = \Pr\left[v^{K+1} > \frac{.02/d}{1 + .02/d}\right]$$

$$= \Pr\left[K < \underbrace{\frac{\log\left(\frac{.02/d}{1+.02/d}\right)}{-\delta} - 1}_{22.04684}\right]$$

$$\simeq \Pr[K \leq 22] = {}_{23}q_{50}$$

$$= 1 - {}_{23}p_{50}$$

$$= 1 - \frac{\ell_{73}}{\ell_{50}} = 1 - \frac{5920394}{8950901}$$

$$= \underline{\underline{0.3385701}}$$

About 0.34 probability of a positive loss-at-issue

Question No. 12:

Suppose you are given:

- $e_x = 30.0$
- $e_{x+1} = 30.6$
- $e_{x+2} = 32.9$

Calculate the probability that (x) will survive the next year but dies the following year.

Apply the recursion $e_x = p_x(1+e_{x+1})$

so that $p_x = e_x/(1+e_{x+1})$

$$p_x = \frac{e_x}{1+e_{x+1}} = \frac{30}{1+30.6} = \frac{30}{31.6}$$

$$p_{x+1} = \frac{e_{x+1}}{1+e_{x+2}} = \frac{30.6}{1+32.9} = \frac{30.6}{33.9}$$

$$_{1|}q_x = {}_1p_x\, q_{x+1} = p_x(1-p_{x+1})$$

$$= \frac{30}{31.6}\left(1-\frac{30.6}{33.9}\right) = \frac{30}{31.6}\left(\frac{3.3}{33.9}\right)$$

$$= .09241627$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK