MATH 3630
Actuarial Mathematics I
Final Examination
Monday, 13 December 2010
Time Allowed: 2 hours (6:00 - 8:00 pm)
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** SUGGESTED SOLUTIONS

- There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.
- Good luck.
- Have a Happy Christmas and a Merry New Year!

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	$\div$ 120	

Question No. 1:

For a fully continuous whole life insurance of \$10 issued to (x), you are given:

- The annual benefit premium is 0.20.
- $\bar{a}_x = 18.75$
- ${}^2\bar{a}_x = 10.00$
- $\delta = 4\%$

Calculate the variance of the loss-at-issue random variable.

$$\bar{a}_{\overline{T|}} = \frac{1-v^T}{\delta}$$

$$L = \text{loss-at-issue} = 10v^T - 0.2\bar{a}_{\overline{T|}}$$

$$= \left(10 + \frac{0.2}{.04}\right)v^T - \frac{0.2}{.04} = 15v^T - 5$$

$$Var(L) = (15)^2 \, Var(v^T)$$

$$= 225\left({}^2\bar{A}_x - \bar{A}_x^2\right)$$

where $\bar{A}_x = 1 - \delta\bar{a}_x = 1 - .04(18.75) = 0.25$

${}^2\bar{A}_x = 1 - 2\delta\,{}^2\bar{a}_x = 1 - .08(10.00) = 0.20$

$$= 225(0.20 - .25^2)$$

$$= \underline{\underline{30.9375}}$$

Question No. 2:

Barbie Dahl and Bill Board are two actuaries who use the same mortality table to price a fully discrete two-year endowment insurance of 100 on (40). You are given:

- Barbie calculates level annual benefit premiums of 1.15.
- Bill calculates non-level benefit premiums of 0.82 for the first year, and p for the second year
- Interest rate i is 8%.
- Both actuaries calculate benefit premiums based on the equivalence principle.

Calculate p.

Barbie: $\pi = 100 \dfrac{\overbrace{A_{40:\overline{2|}}}^{1-d\,\ddot{a}_{40:\overline{2|}}}}{\ddot{a}_{40:\overline{2|}}} = \dfrac{100}{\ddot{a}_{40:\overline{2|}}} - 100\overbrace{d}^{.08/1.08} = 1.15$

$\Rightarrow \ddot{a}_{40:\overline{2|}} = \dfrac{100}{100\left(\frac{.08}{1.08}\right) + 1.15}$

$= 11.68578$

Bill:

$$\begin{array}{cc} .82 & p \\ \hline 40 & 41 \end{array}$$

$\ddot{a}_{40:\overline{2|}} = 1 + v p_x$

$.82 + p \cdot \underbrace{v p_{40}}_{(\ddot{a}_{40:\overline{2|}} - 1)} = 100 A_{40:\overline{2|}}$

$= 100(1 - d\, \ddot{a}_{40:\overline{2|}})$

$\ddot{a}_{40:\overline{2|}}$: 11.68578; d: $\frac{.08}{1.08}$; $\ddot{a}_{40:\overline{2|}}$: 11.68578

$$p = \frac{100\left(1 - \frac{.08}{1.08}(11.68578)\right) - .82}{10.68578}$$

$= \underline{\underline{1.180882}}$

Question No. 3:

For a special whole life insurance on (30) payable at the end of the year of death, you are given:

- The benefit amount is

  100 if death occurs in the first 10 years,

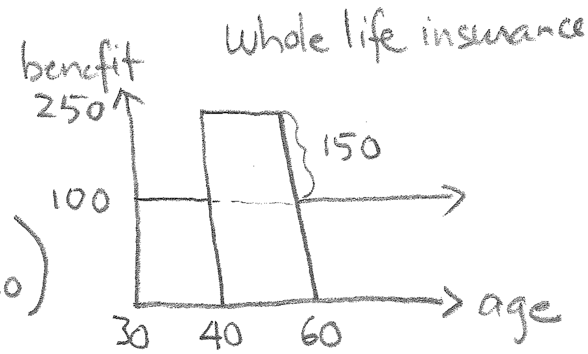
  250 if death occurs the following 20 years, and

  100 if death occurs thereafter.

- The annual benefit premium is level, payable at the beginning of each year that (30) is alive, and is payable only for 20 years.

Mortality follows the *Illustrative Life Table* with $i = 6\%$.

Calculate the amount of the annual benefit premium.



APV(benefits)

$$= 100 A_{30} + 150 \left({}_{10}E_{30} A_{40} - {}_{30}E_{30} A_{60} \right)$$

$$= 100(.10248) + 150(.54733(.16132) - (.54733)(.27414)(.36913))$$

$$= 15.18437$$

$$\text{APV(premiums)} = \pi \ddot{a}_{30:\overline{20}|} = \pi \left(\ddot{a}_{30} - {}_{20}E_{30} \ddot{a}_{50} \right)$$

$$= \pi (\underbrace{15.8561 - .29374(13.2668)}_{11.95911})$$

Thus the annual benefit premium is

$$\pi = \frac{15.18437}{11.95911} = \underline{\underline{1.269691}}$$

Question No. 4:

For a continuous whole life insurance issued to (40), you are given:

- The death benefit at time t is $b_t = 10e^{0.02t}$ for $t > 0$.
- Mortality follows deMoivre's law with $\omega = 100$.
- $\delta = 4\%$

Calculate the actuarial present value for this insurance.

de Moivre's with $\omega = 100 \Rightarrow T_{40} \sim$ Uniform on $(0, 60)$

$$\Rightarrow f_{T_{40}}(t) = \frac{1}{60}, \quad 0 < t < 60$$

Thus,

$$APV(\text{insurance}) = \int_0^{60} b_t v^t f_{T_{40}}(t)\,dt$$

$$= \frac{10}{60} \int_0^{60} \underbrace{e^{.02t} e^{-.04t}}_{e^{-.02t}} dt$$

$$= \frac{1}{6} \cdot \frac{1}{.02}\left(1 - e^{-.02(60)}\right)$$

$$= \underline{\underline{5.823382}}$$

Question No. 5:

For a life (30), you are given:

- $d = 0.065$
- Future lifetime of (30) follows an Exponential distribution with force mortality of 0.03.

Calculate $\ddot{a}_{30:\overline{10|}}$ and <u>interpret this value</u>.

$d = .065 \Rightarrow v = 1 - .065 = .935$

Exponential $\Rightarrow$ constant force $\Rightarrow {}_kp_{30} = e^{-\mu k} = e^{-.03k}$

Thus,

$$\ddot{a}_{30:\overline{10|}} = \sum_{k=0}^{9} v^k \, {}_kp_{30} = \sum_{k=0}^{9} \big(\underbrace{.935 e^{-.03}}_{.9073666}\big)^k$$

$$= \frac{1 - (.9073666)^{10}}{1 - .9073666}$$

$$= \underline{\underline{6.711481}}$$

This is the actuarial present value of an annuity that pays \$1 at the beginning of each year (30) is alive for the next 10 years.

Question No. 6:

For a group of 100 individuals who are all age x, you are given:

- 40 are males (m) and the rest are females (f).
- The following extract from a mortality table:

k	q^m_{x+k}	q^f_{x+k}
0	0.10	0.05
1	0.20	0.10
2	0.30	0.15

where the superscripts m and f refer to male and female, respectively.

Calculate the probability that an individual randomly chosen from this group will die between the ages of $x+2$ and $x+3$.

For male, the probability is ${}_2p^m_x \, q^m_{x+2} = (1-.1)(1-.2)(.3)$
$= .216$

For female, the probability is ${}_2p^f_x \, q^f_{x+2} = (1-.05)(1-.10)(.15)$
$= .12825$

Thus, for a randomly chosen individual, the probability is

$$\frac{40}{100}(.216) + \frac{60}{100}(.12825)$$

$$= \underline{\underline{0.16335}}$$

Question No. 7:

The Vis-ta-vie Life Insurance Company has decided to sell fully discrete whole life insurance of 1,000 to individuals age 50, something it has not done before. The actuaries of Vis-ta-vie have determined that:

- For each policy, the annual benefit premium will be 20.
- Mortality is expected to follow the *Illustrative Life Table* and premiums are calculated at $i = 6\%$.
- All the policyholders will have independent future lifetimes.

Calculate the smallest number of policies the company must sell so that the probability of a positive loss at issue, on the aggregate, does not exceed 0.10. Use the Normal approximation.

Let n be the number of policies to sell so that the aggregate loss-at-issue can be expressed as

$$L = L_1 + L_2 + \cdots + L_n$$

where

$$L_i = 1000 v^{K+1} - 20 \ddot{a}_{\overline{K+1}|} \quad \frac{1-v^{K+1}}{d}$$

$$= \left(1000 + \frac{20}{d}\right) v^{K+1} - \frac{20}{d}, \quad i = 1, 2, \ldots, n$$

$$E(L_i) = \left(1000 + \frac{20}{d}\right) \underset{.24905}{A_{50}} - \frac{20}{d} = -16.28567 \qquad d \to .06/1.06$$

$$Var(L_i) = \left(1000 + \frac{20}{d}\right)^2 \left(\underset{.09476}{{}^2A_{50}} - \underset{.24905^2}{A_{50}^2} \right)$$

$$= 59952.86$$

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$$P(L > 0) \leq 0.10 \Rightarrow P\Big(Z > \underbrace{\frac{16.28567\,n}{\sqrt{59952.86\,n}}}_{1.28}\Big) \leq 0.10$$

$$\Rightarrow \frac{16.28567\,\cancel{n}^{\sqrt{n}}}{\sqrt{59952.86\,\cancel{n}}} \geq 1.28$$

$$\sqrt{n} \geq \frac{1.28\sqrt{59952.86}}{16.28567} = 19.24462$$

$$n \geq 370.3555$$

Company must sell at least 371 policies!

Question No. 8:

A life insurance company has a portfolio of (discrete) whole life annuities-due. Its policyholders consist of non-smokers and smokers whose forces of mortality are, respectively, given by:

$$\mu^{\text{ns}}_{x+t} = 0.02 \ \text{ for } t > 0,$$

and

$$\mu^{\text{s}}_{x+t} = 0.05 \ \text{ for } t > 0.$$

For the same issue age (x), the actuarial present value of B payable annually to a smoker is equal to the actuarial present value of 2 payable annually to a non-smoker. Suppose $\delta = 5\%$.

Calculate the value of B.

$$\text{APV(nonsmoker annuity)} = \sum_{k=0}^{\infty} 2 e^{-.05k} e^{-.02k}$$

$$= 2 \sum_{k=0}^{\infty} e^{-.07k} = \frac{2}{1-e^{-.07}}$$

$$\text{APV(smoker annuity)} = \sum_{k=0}^{\infty} B e^{-.05k} e^{-.05k}$$

$$= B \sum_{k=0}^{\infty} e^{-.10k} = \frac{B}{1-e^{-.10}}$$

The two are actuarially equivalent so that

$$B = \frac{2(1-e^{-.10})}{1-e^{-.07}} = \underline{2.815204}$$

The smoker has worse mortality and is expected to have a shorter life span so that its annual benefit is larger than that of the non-smoker.

Question No. 9:

For a fully continuous whole life insurance of \$1 issued to (x), you are given:

- L_1 is the loss-at-issue random variable if the level benefit premium is determined according to the equivalence principle.
- L_2 is the loss-at-issue random variable if the level benefit premium is 120% of that determined according to the equivalence principle.
- $\bar{A}_x = 0.40$ and $\bar{a}_x = 10$ $\Rightarrow \delta = \dfrac{1-\bar{A}_x}{\bar{a}_x} = \dfrac{1-.40}{10} = .06$
- $\mathrm{Var}(L_1) = 0.25$

Calculate $\mathrm{Var}(L_2)$.

$$L_1 = v^T - \pi \bar{a}_{\overline{T|}} = \left(1+\frac{\pi}{\delta}\right)v^T - \frac{\pi}{\delta} \text{ where } \pi = \frac{\bar{A}_x}{\bar{a}_x} = \frac{.4}{10} = .04$$

$$\mathrm{Var}(L_1) = \left(1+\frac{.04}{.06}\right)^2 \mathrm{Var}(v^T) = \left(\frac{5}{3}\right)^2 \mathrm{Var}(v^T) = \frac{1}{4}$$

$$\Rightarrow \mathrm{Var}(v^T) = \frac{1}{4}\left(\frac{3}{5}\right)^2 = .09$$

Thus,

$$L_2 = \left(1+\frac{1.2\pi}{\delta}\right)v^T - \frac{1.2\pi}{\delta}$$

So that

$$\mathrm{Var}(L_2) = \left(1+\frac{1.2(.04)}{.06}\right)^2 \mathrm{Var}(v^T)$$

$$= (1.8)^2(.09) = \underline{\underline{.2916}}$$

Question No. 10:

Assume the following:

- Mortality follows the *Illustrative Life Table.*
- Interest rate is $i = 10\%$

Calculate $P_{45:\overline{2|}}$ and <u>interpret this value</u>.

$$P_{45:\overline{2|}} = \frac{A_{45:\overline{2|}}}{\ddot{a}_{45:\overline{2|}}} = \frac{1}{\ddot{a}_{45:\overline{2|}}} - d \quad \rightarrow .10/1.10$$

$$\ddot{a}_{45:\overline{2|}} = 1 + v\,p_{45} = 1 + \frac{1}{1.1}\left(1 - \frac{4}{1000}\right) = 1.905455$$

$$P_{45:\overline{2|}} = \frac{1}{1.905455} - \frac{.10}{1.10} = \underline{\underline{0.4339}}$$

This gives the annual premium for a policy that pays \$1 if you die within the next two years and \$1 if you survive the next two years.

Question No. 11:

A whole life insurance contract is issued to (45). You are given:

- Benefit premiums are level and payable at the beginning of each year for 20 years only.
- The death benefit is 10, plus the return of all benefit premiums paid without interest, payable at the end of the year of death.

Give a correct expression for the loss-at-issue random variable associated with this contract. If necessary, define symbols used.

Let π = annual benefit premium

and K = curtate future lifetime of (45)

loss at issue is $L = b_{K+1} v_{K+1} - \pi Y$

where $b_{K+1} = \begin{cases} 10 + (K+1)\pi, & \text{for } K = 0, 1, \ldots, 19 \\ 10 + 20\pi, & \text{for } K = 20, 21, \ldots \end{cases}$

$v_{K+1} = v^{K+1}$, for $K = 0, 1, \ldots$,

$Y = \begin{cases} \ddot{a}_{\overline{K+1|}}, & \text{for } K = 0, 1, \ldots, 19 \\ \ddot{a}_{\overline{20|}}, & \text{for } K = 20, 21, \ldots \end{cases}$

Question No. 12:

For a semi-continuous whole life insurance issued to (30), you are given:

- The death benefit is \$1 payable at the moment of death.
- Benefit premiums are level, payable at the beginning of each quarter of each year for 10 years.
- Deaths are uniformly distributed over each year of age.
- $i = 0.03,\ \alpha(4) = 1.000068,$ and $\beta(4) = 0.379653$
- $\ddot{a}_{30} = 27.75$
- $\ddot{a}_{40} = 21.68$
- ${}_{10}E_{30} = 0.73756$

Calculate the amount of the quarterly premium.

Let $P =$ quarterly premium

$$APV(\text{benefit}) = \bar{A}_{30} = \frac{i}{\delta} A_{30} = \frac{i}{\delta}(1 - d\ddot{a}_{30})$$

$$= \frac{0.03}{\log 1.03}\left(1 - \frac{.03}{1.03}(27.75)\right) = 0.1946096$$

$$APV(\text{Premiums}) = 4P\ddot{a}^{(4)}_{30:\overline{10}|} = 4P\left(\ddot{a}^{(4)}_{30} - {}_{10}E_{30}\,\ddot{a}^{(4)}_{40}\right)$$

$$\ddot{a}^{(4)}_{30} = \underset{1.000068}{\alpha(4)}\ \underset{27.75}{\ddot{a}_{30}} - \underset{.379653}{\beta(4)} = 27.37223$$

$$\ddot{a}^{(4)}_{40} = \alpha(4)\ddot{a}_{40} - \beta(4) = 21.30182$$

$$\therefore P = \frac{0.1946096}{4\underbrace{(27.37223 - .73756(21.30182))}_{46.64345}} = \underline{\underline{.004172282}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK