# MATH 3630
# Actuarial Mathematics I
## Final Examination
Monday, 14 December 2009, Monteith 303
Time Allowed: 2 hours (6:00 – 8:00 PM)
Total Marks: 130 points

**Please write your name and student number at the spaces provided:**

**Name:** SOLUTIONS **Student ID:** EMIL

- There are thirteen (13) written-answer questions here and you are to answer all ~~twelve~~ thirteen. Each question is worth 10 points. Your final mark will be divided by 130 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.
- Good luck.
- Wishing you a Merry Christmas and a Prosperous New Year!

| Question | Worth | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| 11 | 10 | |
| 12 | 10 | |
| 13 | 10 | |
| Total | 130 | |
| % | ÷ 130 | |

**Question No. 1:**

For whole life annuity-due policies on (65) and (66), you are given:

- The annuity payments of \$10 are made annually.
- The interest rate is 0.04 for year 2009 and 0.05 for years 2010 and thereafter.
- $q_{65} = 0.015$ and $q_{66} = 0.016$
- The actuarial present value of the life annuity on (66) is 120 as of the beginning of year 2009.

Calculate the actuarial present value of the life annuity on (65) as of the beginning of year 2009.

Let $\ddot{a}$ be values at 5% in all years

$\ddot{a}^*$ be values at 4% in 1st year, and 5% thereafter

$$\ddot{a}^*_{66} = 1 + v p_{66} \ddot{a}_{67} = \underbrace{1 + \frac{1}{1.04}(1-.016)\ddot{a}_{67} = \frac{120}{10}}$$

$$\Rightarrow \ddot{a}_{67} = \frac{(12-1)(1.04)}{1-.016} = 11.62602$$

$$\ddot{a}_{66} = 1 + v p_{66} \ddot{a}_{67} = 1 + \frac{1}{1.05}(1-.016)\ddot{a}_{67}$$

$$= 1 + \frac{.984}{1.05}(11.62602) = 11.89524$$

$$\therefore \ddot{a}^*_{65} = 1 + v p_{65} \ddot{a}_{66} = 1 + \frac{1}{1.04}(1-.015)(11.89524)$$

$$= \underline{12.26616}$$

the APV of the life annuity on (65) as of b.o.y. 2009

multiply by 10

$10\ddot{a}^*_{65} = 122.6616$

**Question No. 2:**

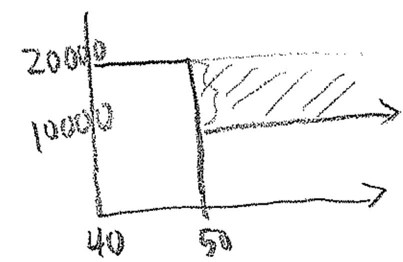
For a special whole life insurance policy issued to (40), the death benefits are:

- payable at the end of the year of death, and
- \$20,000 if death occurs in the first 10 years, and \$10,000 if death thereafter.

The expense-loaded premium, payable annually, is \$697.93 based on the following expenses:

- sales commission is 30% of the expense-loaded premium in the first year, and 5% thereafter;
- acquisition expense is \$80 per \$1,000 of benefit in the first year and maintenance expenses are \$30 per \$1,000 of benefit for years 2 and thereafter; and
- death settlement expense is \$$K$.

death benefit




Mortality follows the *Illustrative Life Table* with $i = 6\%$.

Calculate the value of $K$.

$$APV(FP) = APV(FB) + APV(FE)$$

$$697.93\,\ddot{a}_{40} = 20{,}000\,A_{40} - 10{,}000\,{}_{10}E_{40}\,A_{50}$$
$$+ .05(697.93)\,\ddot{a}_{40} + .25(697.93)$$
$$+ 30\left(20\ddot{a}_{40} - 10\,{}_{10}E_{40}\,\ddot{a}_{50}\right) + 50(20)$$
$$+ K A_{40}$$

Substitute $\ddot{a}_{40} = 14.8166$ $\quad A_{40} = .16132$ $\quad {}_{10}E_{40} = .53667$

$\ddot{a}_{50} = 13.2668$ $\quad A_{50} = .24905$

$$\therefore K = \frac{697.93(.95)(14.8166) - .25(697.93) - [20{,}000(.16132) - 10{,}000(.53667)(.24905)] - 1000 - [600(14.8166) - 300(.53667)(13.2668)]}{.16132}$$

$$= 34.74063 \approx \underline{\underline{35}}$$

**Question No. 3:**

A whole life insurance is issued to (20) with a benefit of \$100 to be paid at the moment of death.
The force of interest is 10% and mortality is based on the following model:

$$\ell_x = 1000\left(1 - \frac{x}{100}\right)^{1/2}, \quad \text{for } x \geq 0.$$

For this insurance, denote the present value of the benefit random variable by Z.

Calculate the probability that Z will exceed \$25.

$$Z = 100\, e^{-.10 T_{20}}$$

$$P(Z > 25) = P\left(100 e^{-.10 T_{20}} > 25\right) = P\left(T_{20} < 10 \log 4\right)$$

$$P(T_{20} > t) = \frac{S_x(20+t)}{S_x(20)} = \frac{\left(1 - \frac{20+t}{100}\right)^{1/2}}{\left(1 - \frac{20}{100}\right)^{1/2}} = \left(\frac{80-t}{80}\right)^{1/2}$$

$$\therefore\ P(T_{20} < 10\log 4) = 1 - P(T_{20} > 10\log 4)$$

$$= 1 - \left(1 - \frac{10\log 4}{80}\right)^{1/2}$$

$$= .090723$$

**Question No. 4:**

You are given the following:

- ${}_{20}V_{20} = 0.082$
- ${}_{30}V_{20} = 0.168$
- ${}_{20}V_{30} = 0.142$ $\Rightarrow 1 - \ddot{a}_{30}/\ddot{a}_{20}$

Calculate ${}_{10}V_{20}$ and interpret this value.

Recall ${}_tV_x = 1 - \ddot{a}_{x+t}/\ddot{a}_x$

$${}_{30}V_{20} = 1 - \ddot{a}_{50}/\ddot{a}_{20} = .168 \Rightarrow \frac{\ddot{a}_{50}}{\ddot{a}_{20}} = 1 - .168 = .832$$

$${}_{20}V_{30} = 1 - \ddot{a}_{50}/\ddot{a}_{30} = .142 \Rightarrow \frac{\ddot{a}_{50}}{\ddot{a}_{30}} = 1 - .142 = .858$$

$$\frac{\ddot{a}_{50}/\ddot{a}_{20}}{\ddot{a}_{50}/\ddot{a}_{30}} = \frac{\ddot{a}_{30}}{\ddot{a}_{20}} = \frac{.832}{.858}$$

$$\therefore \quad {}_{10}V_{20} = 1 - \frac{.832}{.858} = .030$$

$\downarrow$

the benefit reserve at year 10 for a whole life insurance of \$1 issued to (20)

**Question No. 5:**

For a fully discrete whole life insurance of $1 on (35), you are given:

- If death occurs during the first 30 years, the single benefit premium will be returned without interest at the end of the year of death.
- $\ddot{a}_{65} = 10.12$
- ${}_{30}E_{35} = 0.18$
- $A_{35:\overline{30|}} = 0.25$
- $i = 7\%$

let this be $\pi$

Calculate the single benefit premium.

$$APV(\text{Premium}) = APV(\text{Benefits})$$

$$A^{1}_{35:\overline{30|}} + {}_{30}E_{35} A_{65}$$

$$\pi = \pi A^{1}_{35:\overline{30|}} + A_{35}$$

$$\pi(1 - A^{1}_{35:\overline{30|}}) = A_{35} \Rightarrow \pi = \frac{A_{35}}{1 - A^{1}_{35:\overline{30|}}}$$

$$A^{1}_{35:\overline{30|}} = A_{35:\overline{30|}} - {}_{30}E_{35} = .25 - .18 = .07$$

$$A_{65} = 1 - d\ddot{a}_{65} = 1 - \frac{.07}{1.07}(10.12) = 0.34$$

$$\pi = \frac{.07 + .18(.34)}{1 - .07} = \underline{\underline{0.14}}$$

**Question No. 6:**

An insurance company has a portfolio of policies, all issued at the same time, consisting of:

- 100 whole life annuity-due contracts all issued to (65), each with annual payment of \$2, and
- 200 whole life insurance contracts all issued to (50), each with a death benefit of \$1 payable at the end of the year of death

Future lifetimes of all policyholders are independent. Interest rate $i = 6\%$ and you are given:

| $x$ | $1000A_x$ | $1000(^2A_x)$ |
|---|---|---|
| 50 | 205.08 | 63.92 |
| 65 | 401.77 | 199.85 |

Using Normal approximation, calculate the amount of the fund needed at issue to be 99% certain of having enough money to pay all the benefits in the portfolio. (Note: the 99th percentile of a standard Normal is 2.326.)

Let $S = Y + Z$ be the aggregate benefit in the portfolio consisting of $Y = \sum_{i=1}^{100} 2Y_i$ and $Z = \sum_{i=1}^{200} Z_i$ where

$$EY_i = \ddot{a}_{65} = \frac{1-A_{65}}{d} = \frac{1-.40177}{.06/1.06} = 10.56873$$

$$VarY_i = \frac{^2A_{65} - (A_{65})^2}{d^2} = \frac{.19985-(.40177)^2}{(.06/1.06)^2} = 11.9947$$

$EY = 200 * EY_i = 2113.746$ and $VarY = 400 * VarY_i = 4797.88$

$EZ_i = A_{50} = .20508$

$VarZ_i = {}^2A_{50} - (A_{50})^2 = .02186$

$\therefore EZ = 200 * EZ_i = 41.016$ and $VarZ = 200 * VarZ_i = 4.37244$

$ES = EY + EZ = 2154.762$

$VarS = VarY + VarZ = 4797.88 + 4.37244 = 4802.253$

$$P(S \le F) = 0.99 \Rightarrow P\left(\frac{S-ES}{\sqrt{VarS}} \le \underbrace{\frac{F-2154.762}{\sqrt{4802.253}}}_{2.326}\right) = .99 \Rightarrow F = 2154.762 + 2.326 * \sqrt{4802.253} = 2,315.95$$

**Question No. 7:**

You are given the following:

- $\ddot{a}_{30} = 21.30$ $\Rightarrow$ $A_{30} = 1 - d\ddot{a}_{30} = 1 - \frac{.035}{1.035}(21.3) = 0.2797101$
- $\ddot{a}_{40} = 20.54$ $\Rightarrow$ $A_{40} = 1 - d\ddot{a}_{40} = 1 - \frac{.035}{1.035}(20.54) = 0.3054106$
- ${}_{10}E_{30} = 0.68$
- $i = 3.5\%$
- Deaths are uniformly distributed over each year of age interval.

Calculate $A^{(4)}_{30:\overline{10|}}$ and interpret this actuarial symbol.

By UDD assumption, $A^{(4)}_{30:\overline{10|}} = \frac{i}{i^{(4)}} A^{1}_{30:\overline{10|}} + {}_{10}E_{30}$

$i^{(4)} = 4[(1.035)^{1/4} - 1] = .03454978$

$A_{30} = A^{1}_{30:\overline{10|}} + {}_{10}E_{30} A_{40}$

$\Rightarrow A^{1}_{30:\overline{10|}} = A_{30} - {}_{10}E_{30} A_{40}$

$= .2797101 - (.68)(.3054106)$

$= .07203092$

$\therefore A^{(4)}_{30:\overline{10|}} = \frac{.035}{.03454978}(.07203092) + .68 = 0.753$

(the APV of a 10-year endowment issued to (30) with death benefit payable at the end of the quarter in the year of death.

**Question No. 8:**

For a 10-year term life insurance policy of \$1 issued to (25), you are given:

- Death benefit is payable at the end of the year of death.
- Level premium is payable at the beginning of each year for 5 years.

Using actuarial symbols you have learned from this course, give both the prospective and the retrospective benefit reserve formulas at the end of 4 years.

Clearly, the annual premium is $_5P^1_{25:\overline{10|}} = A^1_{25:\overline{10|}} / \ddot{a}_{25:\overline{5|}}$

Prospective formula:

$$_4^5V^1_{25:\overline{10|}} = A^1_{29:\overline{6|}} - {_5P^1_{25:\overline{10|}}} \underbrace{\ddot{a}_{29:\overline{1|}}}_{=1} = A^1_{29:\overline{6|}} - {_5P^1_{25:\overline{10|}}}$$

Retrospective formula:

$$_4^5V^1_{25:\overline{10|}} = \frac{_5P^1_{25:\overline{10|}} \ddot{a}_{25:\overline{4|}}}{_4E_{25}} - \frac{A^1_{25:\overline{4|}}}{_4E_{25}}$$

**Question No. 9:**

A life insurer issues a fully continuous whole life policy of $1 to (30). You are given:

- Force of mortality is constant at $\mu$.
- Force of interest is constant with $\delta = 4\mu$.
- Premium is determined according to the actuarial equivalence principle.

Determine the probability that the insurance company will make a profit.

Premium is $\bar{P}(\bar{A}_{30}) = \mu = \delta/4$

Loss at issue $= {}_0L = v^{T_{30}} - \bar{P}(\bar{A}_{30})\bar{a}_{\overline{T_{30}}|}$

$$= e^{-\delta T_{30}} - \frac{\cancel{\delta}}{4}\left(\frac{1-e^{-\delta T_{30}}}{\cancel{\delta}}\right)$$

$$= \frac{5}{4}e^{-\delta T_{30}} - \frac{1}{4}$$

$$P({}_0L < 0) = P\left(\frac{5}{4}e^{-\delta T_{30}} < \frac{1}{4}\right)$$

$$= P\left(e^{-\delta T_{30}} < \frac{1}{5}\right) = P\left(T_{30} > \frac{1}{\delta}\log 5\right)$$

$$= \exp\left(-\left(\frac{1}{\cancel{\delta}}\log 5\right)\frac{\cancel{\delta}}{4}\right)$$

$$= 5^{-1/4}$$

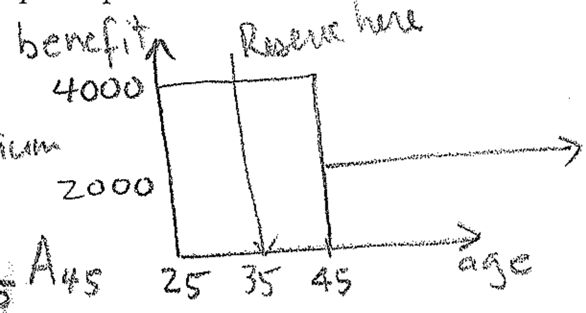
$$= .6687403$$

**Question No. 10:**

For a special whole life policy issued at age 25, you are given:

- Death benefit is payable at the end of the year of death, with benefit amount equal to \$4,000 if death is within the first 20 years and \$2,000 if death is thereafter.
- Mortality follows the *Illustrative Life Table* with $i = 6\%$.
- The level annual benefit premium is payable at the beginning of each year and is determined according to the actuarial equivalence principle.

Calculate the 10th year benefit reserve.




First calculate the benefit premium

$$P\ddot{a}_{25} = 4000A_{25} - 2000\,{}_{20}E_{25}A_{45}$$

$$16.2242 \qquad 4(81.65) - 2(298.73)(.20120)$$

$$P = 12.72118$$

Easier to use prospective formula

$${}_{10}V = APV(FB_{10}) - APV(FP_{10})$$

$$= \left(4000A_{35} - \underbrace{2000\,{}_{10}E_{35}A_{45}}\right) - P\ddot{a}_{35}$$

$$\left(4(128.72) - 2(543.18)(.20120)\right) - 12.72118\,(15.3926)$$

$$= \underline{\underline{100.4923}}$$

**Question No. 11:**

For two mortality assumptions labeled A and B, their forces of mortality are related as follows:

$$\mu_x^{\mathrm{A}} = \beta \mu_x^{\mathrm{B}}, \text{ for some } \beta > 0, \text{ for all } x \geq 0.$$

You are given:

- ${}_{10}q_{30}^{\mathrm{A}} = 0.271$
- ${}_{10}q_{30}^{\mathrm{B}} = 0.190$
- ${}_{20}q_{30}^{\mathrm{B}} = 0.360$

Calculate ${}_{20}q_{30}^{\mathrm{A}}$.

$${}_tp_x^A = \exp\left(-\int_0^t \mu_{x+s}^A \, ds\right) = \exp\left(-\beta \int_0^t \mu_{x+s}^B \, ds\right) = \left({}_tp_x^B\right)^\beta$$

$$\therefore \; {}_{10}p_{30}^A = .729 = \left({}_{10}p_{30}^B\right)^\beta = (.810)^\beta$$

$$\Rightarrow \beta = \log(.729)/\log(.810) = 1.5$$

$${}_{20}p_{30}^A = \left({}_{20}p_{30}^B\right)^{1.5} = (1-.360)^{1.5}$$

$$= .512$$

$${}_{20}q_{30}^A = 1 - .512$$

$$= \underline{\underline{0.488}}$$

**Question No. 12:**

You are given:

- $q_x = 0.04$, for all $x \geq 0$
- $A_x = \frac{1}{6}$

Calculate ${}^2A_x$.

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\infty} (.96v)^k = \frac{1}{1-.96v}$$

$$A_x = 1 - d\ddot{a}_x = 1 - \frac{1-v}{1-.96v} = \frac{1}{6} \Rightarrow \frac{1-v}{1-.96v} = \frac{5}{6}$$

$$\Rightarrow v = 1/1.2$$

$${}^2\ddot{a}_x = \frac{1}{1-.96v^2} = \frac{1}{1-.96/1.2^2} = 3$$

$${}^2A_x = 1-(1-v^2)\,{}^2\ddot{a}_x = 1-\left(1-\left(\frac{1}{1.2}\right)^2\right)*3$$

$$= \underline{\underline{0.08333}} \quad \frac{1}{12}!$$

**Question No. 13:**

You are given:

$$\mu_{40+t} = 0.02, \quad \text{for all } t > 0$$

and

$$\delta_t = \begin{cases} 0.05, & \text{for } 0 < t \le 10 \\ 0.10, & \text{for } t > 10 \end{cases}.$$

Calculate $\bar{a}_{40}$.

$$\bar{a}_{40} = \int_0^\infty v^t \, {}_tp_{40} \, dt$$

$$= \int_0^{10} e^{-.05t} e^{-.02t} dt + \int_0^\infty \underbrace{e^{-.7}}_{v^{10}\,{}_{10}p_{40}} \cdot e^{-.10t} e^{-.02t} dt$$

$$= \frac{1}{.07}(1 - e^{-.7}) + e^{-.7} \cdot \frac{1}{.12}$$

$$= \underline{\underline{11.32985}}$$