MATH 3630
Actuarial Mathematics I
Class Test 1
Wednesday, 30 September 2009
Time Allowed: 1 hour
Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: SOLUTIONS Student ID: EMIL

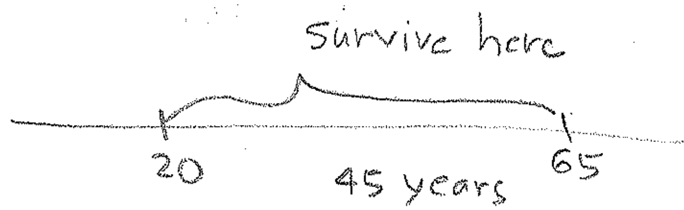
- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.

Question No. 1:

Suppose you are given that

$$\ell_x = 10000(100-x)^2, \text{ for } 0 \leq x \leq 100.$$

Calculate the probability that a person now age 20 will reach retirement age 65.



We want $\ _{45}p_{20} = \dfrac{\ell_{65}}{\ell_{20}}$

$$= \frac{\cancel{10000}\,(100-65)^2}{\cancel{10000}\,(100-20)^2}$$

$$= \frac{35^2}{80^2} = \left(\frac{35}{80}\right)^2 = \left(\frac{7}{16}\right)^2$$

$$= \underline{\underline{0.1914}}$$

Question No. 2:

Two friends, Daniel and Brian, are of the same age x. Denote their respective future lifetimes by T_x^{D} and T_x^{B}.

You are given:

- Daniel has a constant force of mortality $\mu_x^{\mathrm{D}} = 0.010$ for all $x > 0$.
- Brian has a constant force of mortality $\mu_x^{\mathrm{B}} = 0.015$ for all $x > 0$.

Calculate the probability that Daniel's future lifetime will exceed Brian's expected future lifetime.

Since T_x^B is constant force, $T_x^B \sim$ Exponential with mean $E(T_x^B) = \frac{1}{\mu_x^B} = \frac{1}{.015}$

The required probability is

$$P(T_x^D > E(T_x^B)) = P\left(T_x^D > \frac{1}{.015}\right) \qquad T_x^D \sim \text{Exponential}$$

$$= \exp(-.010/.015)$$

$$= e^{-2/3}$$

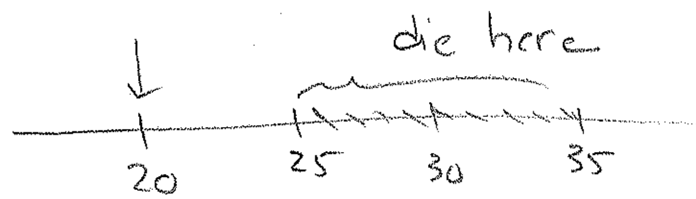
$$= \underline{0.5134}$$

Question No. 3:

You are given the following force of mortality:

$$\mu_x = \begin{cases} 0.02, & \text{for } 0 < x \leq 25 \\ 0.03, & \text{for } x > 25 \end{cases}$$

Calculate the probability that a life (20) will die between the ages of 25 and 35.



$$ {}_{5|10}q_{20} = {}_{5}p_{20} - {}_{15}p_{20} $$

where $\quad {}_{t}p_{20} = e^{-\int_0^t \mu_{20+s}\,ds} \quad$ and $\quad \mu_{20+s} = \begin{cases} .02, & 0 < s \leq 5 \\ .03, & s > 5 \end{cases}$

$$ {}_{5}p_{20} = e^{-\int_0^5 .02\,ds} = e^{-.02(5)} = e^{-.10} $$

$$ {}_{15}p_{20} = e^{-\int_0^5 .02\,ds}\; e^{-\int_5^{15} .03\,ds} $$

$$ = e^{-.10}\; e^{-.30} $$

$$ \therefore\; {}_{5|10}q_{20} = e^{-.10}\left(1 - e^{-.30}\right) $$

$$ = \cancel{0.1260}\;\; 0.2345 $$

Question No. 4:

In a certain population where 3/4 are females and 1/4 are males, the survival function for a female newborn is given by

$$S^f_X(x) = \left(1 - \frac{x}{100}\right)^{1/2}, \text{ for } 0 \le x \le 100,$$

and that for a male newborn is

$$S^m_X(x) = \left(1 - \frac{x}{90}\right)^{1/2}, \text{ for } 0 \le x \le 90.$$

For a randomly selected member of this population, calculate $P(K_{50} = 10)$ where K_{50} is the curtate future lifetime of (50).

death here

50 60 61

Note that $P(K_{50}=10) = {}_{10|}q_{50} = \underbrace{{}_{10}p_{50}\, q_{60}}_{[S_X(60) - S_X(61)] / S_X(50)}$

For female, $P(K^f_{50}=10) = (40^{1/2} - 39^{1/2})/50^{1/2} = .0112510$

For male, $P(K^m_{50}=10) = (30^{1/2} - 29^{1/2})/40^{1/2} = .01455609$

For a randomly selected member,

$$P(K_{50}=10) = \frac{3}{4}P(K^f_{50}=10) + \frac{1}{4}P(K^m_{50}=10)$$

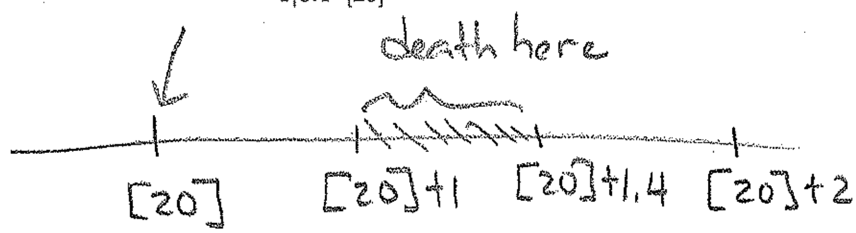
$$= .0121$$

Question No. 5:

Suppose you are given the following select-and-ultimate mortality table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
20	35,600	35,591	35,583	35,572	23
21	35,587	35,576	35,562	35,553	24
22	35,574	35,561	35,547	35,534	25

Assuming UDD between integral ages, calculate ${}_{1|0.4}q_{[20]}$.



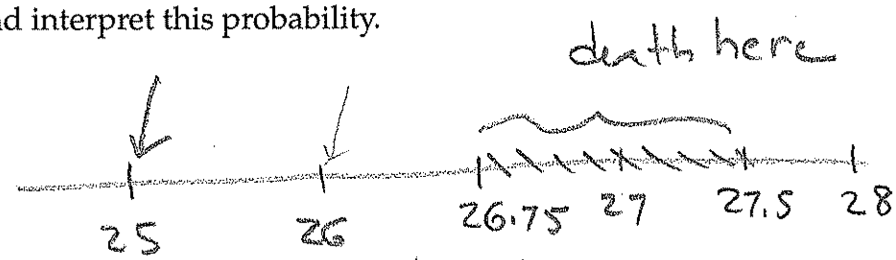
$$
\begin{aligned}
{}_{1|0.4}q_{[20]} &= p_{[20]} \; {}_{0.4}q_{[20]+1} \\
&= p_{[20]} * 0.4 \; q_{[20]+1} \quad \text{by UDD} \\
&= 0.4 \underbrace{\frac{\ell_{[20]+1}}{\ell_{[20]}} * \left(1 - \frac{\ell_{[20]+2}}{\ell_{[20]+1}}\right)}_{\dfrac{\ell_{[20]+1} - \ell_{[20]+2}}{\ell_{[20]}}} \\
&= 0.4 \left(\frac{35591 - 35583}{35600} \right) \\
&= \underline{.00009}
\end{aligned}
$$

Question No. 6:

Assume mortality follows the *Illustrative Life Table*.

Suppose that constant force of mortality assumption holds between integral ages.

Calculate ${}_{1.75|0.75}q_{25}$ and interpret this probability.



For constant force between integral ages,

recall ${}_tp_x = p_x^t, \quad 0 \le t \le 1, \quad x = \text{integer}$

It is easy to see that

$$
\begin{aligned}
{}_{1.75|0.75}q_{25} &= p_{25} * \left({}_{.75}p_{26} - p_{26} * {}_{.5}p_{27} \right) \\
&= p_{25} * \left(p_{26}^{.75} - p_{26} * p_{27}^{.5} \right) \\
&= .99878 * \left(.99873^{.75} - .99875 * .99867^{.5} \right) \\
&= \underline{\underline{.00096}}
\end{aligned}
$$

Question No. 7:

Suppose you are given the following survival function

$$S_X(x) = e^{-.015x}, \text{ for } x > 0.$$

Calculate e_{30} and interpret this value.

Recall $e_x = \sum_{k=1}^{\infty} {}_kp_x \Rightarrow e_{30} = \sum_{k=1}^{\infty} {}_kp_{30}$

where ${}_kp_{30} = \dfrac{S_X(30+k)}{S_X(30)} = \dfrac{e^{-.015(\cancel{30}+k)}}{\cancel{e^{-.015(30)}}} = e^{-.015k}$

Thus,

$$e_{30} = \sum_{k=1}^{\infty} e^{-.015k} = \frac{e^{-.015}}{1-e^{-.015}} = \underline{66.1679}$$

This gives the average number of completed years lived for a person age 30 today.

Question No. 8:

Prove that the following holds:

$$\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_n p_x \cdot \mathring{e}_{x+n}.$$

Now, suppose you are given:

$$\begin{aligned} \mathring{e}_{20} &= 5.00 \\ \mathring{e}_{20:\overline{5}|} &= 3.75 \\ {}_5 p_{20} &= 0.50 \end{aligned}$$

Use the above result to compute $\mathring{e}_{25}$.

$$\mathring{e}_x = \int_0^\infty {}_t p_x \, dt = \underbrace{\int_0^n {}_t p_x \, dt}_{\mathring{e}_{x:\overline{n}|}} + \underbrace{\int_n^\infty {}_t p_x \, dt}_{\text{change variable}}$$

$t^* = t - n$

$dt^* = dt$

$$= \mathring{e}_{x:\overline{n}|} + \int_0^\infty \underbrace{{}_{t^*+n} p_x}_{{}_n p_x \, * \, {}_{t^*} p_{x+n}} \, dt^*$$

$$= \mathring{e}_{x:\overline{n}|} + {}_n p_x \underbrace{\int_0^\infty {}_{t^*} p_{x+n} \, dt^*}_{\mathring{e}_{x+n}}$$

which proves result

Since $\mathring{e}_{20} = \mathring{e}_{20:\overline{5}|} + {}_5 p_{20} \, \mathring{e}_{25}$

$$5 = 3.75 + .5 \, \mathring{e}_{25}$$

$$\frac{1.25}{.5} = 2.5 = \mathring{e}_{25}$$

Question No. 9:

Justin is currently 40 years old and his mortality follows deMoivre's law with $\omega = 100$.

Justin is contemplating taking up rock climbing as a recreational sport in the coming year. His assumed mortality will be adjusted for the coming year only, so that he will instead have a constant force of mortality of 0.04.

Calculate the percentage decrease in Justin's survival probability for the coming year as a result of taking up rock climbing.

Without recreational sport, $T_{40} \sim$ Uniform on $(0,60)$

$$\therefore \; q_{40} = \int_0^1 \frac{1}{60}\,dt = \frac{1}{60} \Rightarrow p_{40} = 59/60$$

With recreational sport, $T_{40} \sim$ Exponential with $\mu = .04$

$$\therefore \; q_{40}^{*} = \int_0^1 .04\,e^{-.04t}\,dt = 1 - e^{-.04}$$

$$\Rightarrow p_{40}^{*} = e^{-.04}$$

Thus,

$$\text{percent decrease} = \frac{p_{40} - p_{40}^{*}}{p_{40}} = \frac{\frac{59}{60} - e^{-.04}}{59/60}$$

$$= \underline{2.29\,\%}$$

Justin's survival probability will drop by 2.29 % as a result of pursuing a dangerous sports activity.

QV corrected 29 Sep 2010 Thanks to JL

Question No. 10:

You are given the survival function:

$$s_X(x) = \left(1 - \frac{x}{\omega}\right)^r, \text{ for } 0 \leq x \leq \omega, \text{ and } r > 0.$$

If $\mu_y = 0.10$ and $\mathring{e}_y = 8.75$ for some $0 < y < \omega$, calculate the value of r.

Given $S_X(x)$, note that $\mu_x = \frac{-d}{dx} \log S_X(x) = \frac{-d}{dx} r \log\left(1 - \frac{x}{\omega}\right)$

$$= \frac{-r}{1-\frac{x}{\omega}}\left(-\frac{1}{\omega}\right) = \frac{r}{\omega - x}$$

and that $\mathring{e}_x = \int_0^{\omega-x} {}_tp_x \, dt = \int_0^{\omega-x} \left(\frac{\omega - x - t}{\omega - x}\right)^r dt$

$$= \frac{-1}{(\omega-x)^r} \frac{(\omega-x-t)^{r+1}}{r+1}\Bigg|_0^{\omega-x} = \frac{\omega - x}{r+1}$$

Thus for some y,

$$\mu_y = \frac{r}{\omega - y} = 0.10 \Rightarrow r = 0.10(\omega - y)$$

$$\mathring{e}_y = \frac{\omega - y}{r+1} = 8.75 \Rightarrow \omega - y = 8.75(r+1)$$

Thus

$$r = 0.10(8.75)(r+1)$$

$$= .875r + .875$$

$$r = \frac{.875}{1 - .875} = \underline{7}.$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK