Sutured Floer homology and Seifert surfaces

Faramarz Vafaee

Michigan State University

April 1st, 2012

Motivation

Motivation

- Every knot in S^3 bounds a Seifert surface

Motivation

- Every knot in S^3 bounds a Seifert surface
- *Genus of K*: Minimum genus taken over all oriented surfaces that K bounds

Motivation

- Every knot in S^3 bounds a Seifert surface
- *Genus of K*: Minimum genus taken over all oriented surfaces that K bounds
- Is a given minimal genus Seifert surface for a knot unique?

Motivation

- Every knot in S^3 bounds a Seifert surface
- *Genus of K*: Minimum genus taken over all oriented surfaces that K bounds
- Is a given minimal genus Seifert surface for a knot unique?
- R and R' are *equivalent* if there is an isotopy of S^3 taking R to R'

Motivation

- Every knot in S^3 bounds a Seifert surface
- *Genus of K*: Minimum genus taken over all oriented surfaces that K bounds
- Is a given minimal genus Seifert surface for a knot unique?
- R and $R^{'}$ are *equivalent* if there is an isotopy of S^3 taking R to $R^{'}$
- Fiberedness of a knot is a sufficient condition for which its minimal genus Seifert surface is unique

Motivation

Motivation

- We find

Motivation

- We find
  - A family of knots with trivial Alexander polynomial

Motivation

- We find
  - A family of knots with trivial Alexander polynomial
  - Construct two non-isotopic Seifert surfaces for each member in our family

Motivation

- We find
  - A family of knots with trivial Alexander polynomial
  - Construct two non-isotopic Seifert surfaces for each member in our family
- Classical methods fail in distinguishing the two Seifert surfaces

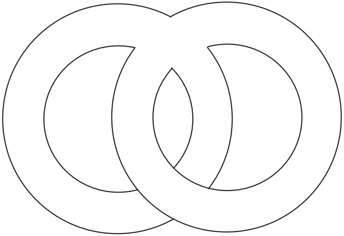
Constructing the knots

Constructing the knots

- Plumb two untwisted annuli

Constructing the knots

- Plumb two untwisted annuli



Constructing the knots

- Plumb two untwisted annuli
- Tie arbitrary nontrivial knots, K_1 and K_2 in the annuli

Constructing the knots

- Plumb two untwisted annuli
- Tie arbitrary nontrivial knots, K_1 and K_2 in the annuli
- Produce some twists in each annulus such that

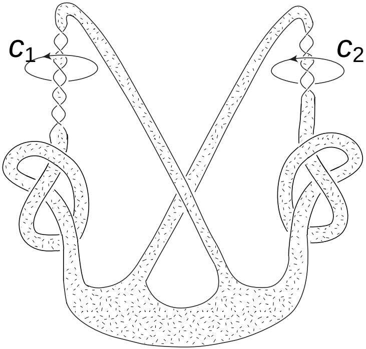
Constructing the knots

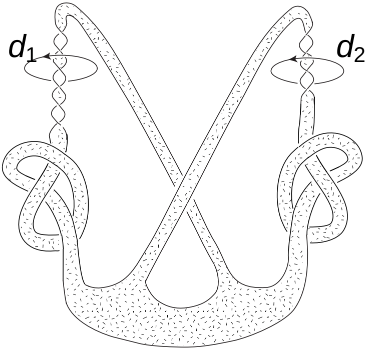
- Plumb two untwisted annuli
- Tie arbitrary nontrivial knots, K_1 and K_2 in the annuli
- Produce some twists in each annulus such that
  - The framing of the first annulus is 0

Constructing the knots

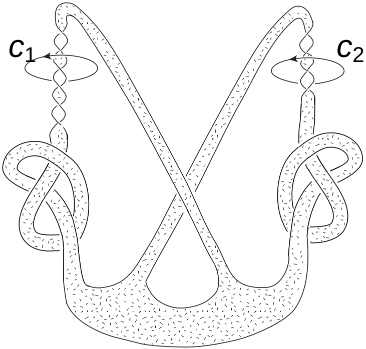
- Plumb two untwisted annuli
- Tie arbitrary nontrivial knots, K_1 and K_2 in the annuli
- Produce some twists in each annulus such that
  - The framing of the first annulus is 0
  - The framing of the other annulus is l, where $l \neq 0$

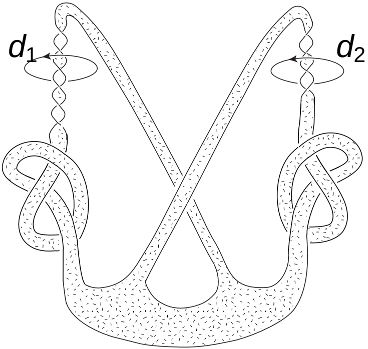
Example





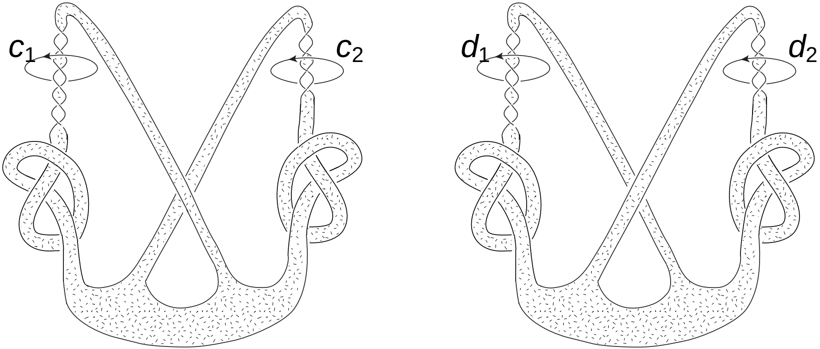
Example





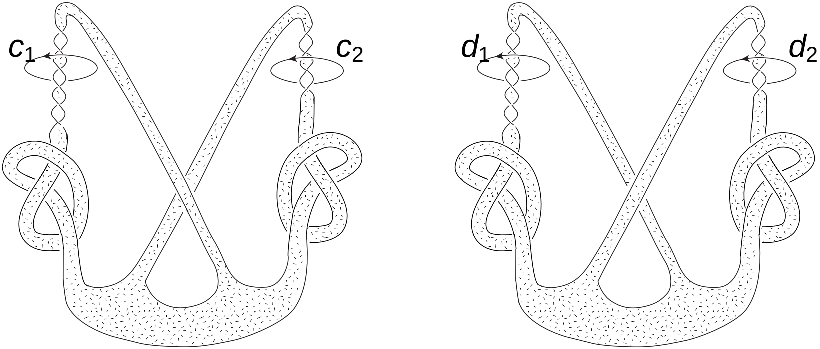
- R and R' two Seifert surfaces

Example



- R and R' two Seifert surfaces
- Both R and R' bounded by $P(K_1, K_2)$

Example

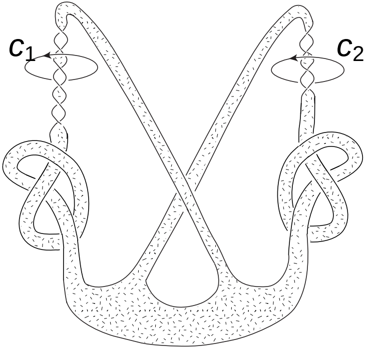


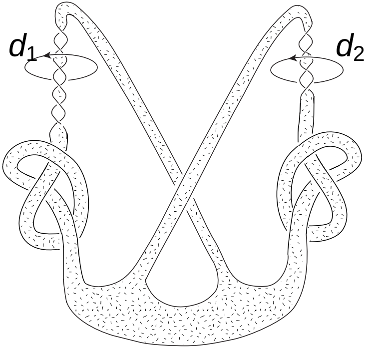
- R and R' two Seifert surfaces
- Both R and R' bounded by $P(K_1, K_2)$
- c_1, c_2, d_1 and d_2 basis elements for H_1 of the complements inside S^3

Main theorem

Theorem(V)

Let $P(K_1, K_2)$ be the knot obtained by plumbing two annuli with arbitrary knots K_1 and K_2 as the following Figure, with framings l and 0, respectively, $l \neq 0$. Changing the plumbing results in the same knot, but two inequivalent Seifert surfaces, R and R'.





Technique

Technique

- The surfaces' complements have a particular structure: Sutured manifold

Technique

- The surfaces' complements have a particular structure: Sutured manifold
- Invariant of sutured manifolds: Sutured Floer Homology(denoted by *SFH*)

Technique

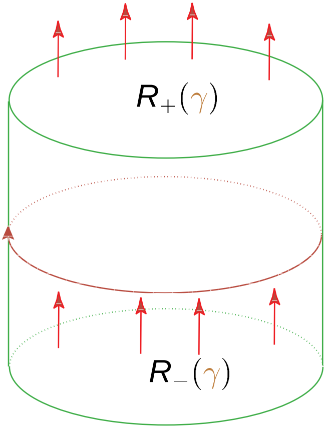
- The surfaces' complements have a particular structure: Sutured manifold
- Invariant of sutured manifolds: Sutured Floer Homology(denoted by *SFH*)
- *SFH* as a Spin^c-graded group can be used to distinguish the surfaces

Technique

- The surfaces' complements have a particular structure: Sutured manifold
- Invariant of sutured manifolds: Sutured Floer Homology(denoted by *SFH*)
- *SFH* as a Spin^c-graded group can be used to distinguish the surfaces
- (SFH+Seifert form) useful to distinguish different Seifert surfaces

Example of a sutured manifold

An example of a sutured manifold



Definition of a sutured manifolds

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$
  - γ consists of annuli $A(\gamma)$ and tori $T(\gamma)$

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$
  - γ consists of annuli $A(\gamma)$ and tori $T(\gamma)$
  - The interior of each component of $A(\gamma)$ contains a suture. The union of sutures: $s(\gamma)$

Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$
  - γ consists of annuli $A(\gamma)$ and tori $T(\gamma)$
  - The interior of each component of $A(\gamma)$ contains a suture. The union of sutures: $s(\gamma)$
- Take $R(\gamma) = \partial M \backslash int(\gamma)$

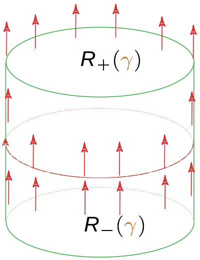
Definition of a sutured manifolds

- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$
  - γ consists of annuli $A(\gamma)$ and tori $T(\gamma)$
  - The interior of each component of $A(\gamma)$ contains a suture. The union of sutures: $s(\gamma)$
- Take $R(\gamma) = \partial M \backslash int(\gamma)$
- $R_+(\gamma)$ ($R_-(\gamma)$), those components of $R(\gamma)$ whose normal vector points out of(into) M

Definition of a sutured manifolds

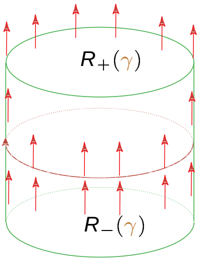
- A *sutured manifold* (M, γ) is a
  - Compact oriented 3-manifold with boundary
  - Together with a set $\gamma \subset \partial M$
  - γ consists of annuli $A(\gamma)$ and tori $T(\gamma)$
  - The interior of each component of $A(\gamma)$ contains a suture. The union of sutures: $s(\gamma)$
- Take $R(\gamma) = \partial M \backslash int(\gamma)$
- $R_+(\gamma)$ ($R_-(\gamma)$), those components of $R(\gamma)$ whose normal vector points out of(into) M
- $S^3(R) = S^3 \backslash int(R \times I)$. Equip this with $\gamma = \partial R \times \{1/2\}$

Sutured manifolds



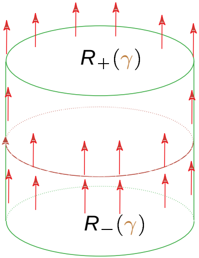
Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M



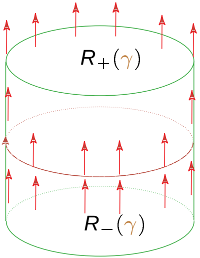
Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M
  - Pointing into M along $R_-(\gamma)$



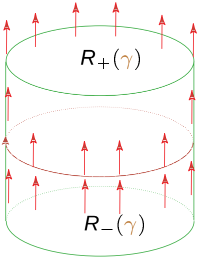
Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M
  - Pointing into M along $R_-(\gamma)$
  - Pointing out of M along $R_+(\gamma)$



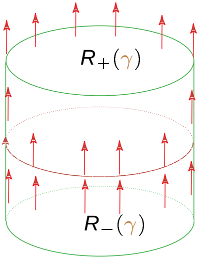
Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M
  - Pointing into M along $R_-(\gamma)$
  - Pointing out of M along $R_+(\gamma)$
  - Restricts to γ to be the gradient of a height function $s(\gamma) \times I \to I$



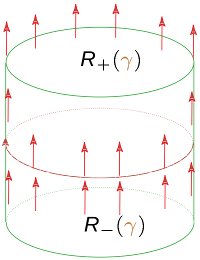
Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M
  - Pointing into M along $R_-(\gamma)$
  - Pointing out of M along $R_+(\gamma)$
  - Restricts to γ to be the gradient of a height function $s(\gamma) \times I \to I$
- The space of such vector fields is contractible



Sutured manifolds

- Let v_0 be a nowhere vanishing vector field on ∂M
  - Pointing into M along $R_-(\gamma)$
  - Pointing out of M along $R_+(\gamma)$
  - Restricts to γ to be the gradient of a height function $s(\gamma) \times I \to I$
- The space of such vector fields is contractible
- It makes sense to fix a representative v_0



Relative Spin^c-structures

Relative Spin^c-structures

- $\underline{Spin^c}(M, \gamma)$: Equivalence classes of nowhere vanishing vector field on M agreeing with v_0 along ∂M

Relative Spin^c-structures

- $\underline{Spin^c}(M, \gamma)$: Equivalence classes of nowhere vanishing vector field on M agreeing with v_0 along ∂M
- $\underline{Spin^c}(M, \gamma)$: An affine space over $H^2(M, \partial M; \mathbb{Z})$

Relative Spin^c-structures

- $\underline{Spin^c}(M, \gamma)$: Equivalence classes of nowhere vanishing vector field on M agreeing with v_0 along ∂M
- $\underline{Spin^c}(M, \gamma)$: An affine space over $H^2(M, \partial M; \mathbb{Z})$
- $\epsilon(\mathfrak{s}_1, \mathfrak{s}_2) = PD^{-1}[\mathfrak{s}_1 - \mathfrak{s}_2]$ for $\mathfrak{s}_1, \mathfrak{s}_2 \in \underline{Spin^c}(M, \gamma)$

Sutured Floer homology

Sutured Floer homology

- Sutured Floer homology splits

$$SFH(M,\gamma) \cong \bigoplus_{\mathfrak{s} \in \underline{Spin}^c(M,\gamma)} SFH(M,\gamma,\mathfrak{s})$$

Sutured Floer homology

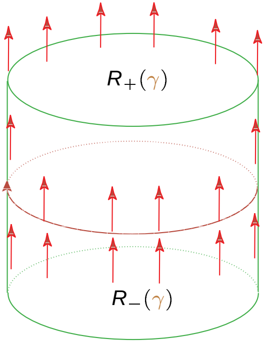
- Sutured Floer homology splits

$$SFH(M,\gamma) \cong \bigoplus_{\mathfrak{s} \in \underline{Spin}^c(M,\gamma)} SFH(M,\gamma,\mathfrak{s})$$

- The *support* of $SFH(M,\gamma)$ is

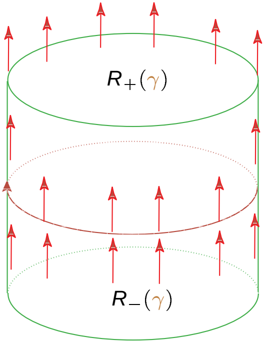
$$S(M,\gamma) = \{\mathfrak{s} \in Spin^c(M,\gamma) : SFH(M,\gamma,\mathfrak{s}) \neq 0\}$$

Relative Euler class



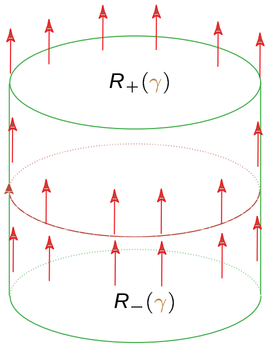
Relative Euler class

- $v_0^{\perp}$ is a trivial vector bundle over ∂M



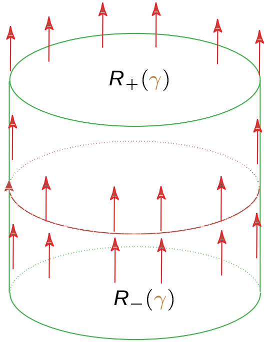
Relative Euler class

- $v_0^\perp$ is a trivial vector bundle over ∂M
- $T(M,\gamma)$ is the set of all trivializations of $v_0^\perp$



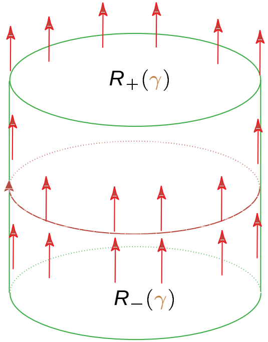
Relative Euler class

- $v_0^{\perp}$ is a trivial vector bundle over ∂M
- $T(M,\gamma)$ is the set of all trivializations of $v_0^{\perp}$
- For a trivialization $\mathfrak{t} \in T(M,\gamma)$



Relative Euler class

- $v_0^\perp$ is a trivial vector bundle over ∂M
- $T(M,\gamma)$ is the set of all trivializations of $v_0^\perp$
- For a trivialization $\mathfrak{t} \in T(M,\gamma)$
  - $c_1(\mathfrak{s},\mathfrak{t})$: The relative Euler class of the vector bundle $v^\perp$ with respect to the trivialization $\mathfrak{t}$



Polytope

Polytope

- Fix a trivialization $\mathfrak{t} \in T(M, \gamma)$

Polytope

- Fix a trivialization $\mathfrak{t} \in T(M, \gamma)$
- $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$

Polytope

- Fix a trivialization $\mathfrak{t} \in T(M, \gamma)$
- $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$
- Define

Polytope

- Fix a trivialization $\mathfrak{t} \in T(M, \gamma)$
- $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$
- Define

$$C(M, \gamma, \mathfrak{t}) = \{c_1(\mathfrak{s}, \mathfrak{t}) : \mathfrak{s} \in S(M, \gamma)\} \subset H^2(M, \partial M; \mathbb{R})$$

Polytope

- Fix a trivialization $\mathfrak{t} \in T(M, \gamma)$
- $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$
- Define

$$C(M, \gamma, \mathfrak{t}) = \{c_1(\mathfrak{s}, \mathfrak{t}) : \mathfrak{s} \in S(M, \gamma)\} \subset H^2(M, \partial M; \mathbb{R})$$

- $P(M, \gamma, \mathfrak{t})$: The polytope obtained as the convex hull of $C(M, \gamma, \mathfrak{t})$ inside $H^2(M, \gamma; \mathbb{R})$

The surface R before plumbing

The surface R before plumbing

The surface R before plumbing

- Knotted annuli with oriented sutures, $A(K_1)$ and $A(K_2)$

The surface R before plumbing

- Knotted annuli with oriented sutures, $A(K_1)$ and $A(K_2)$
- The complement of each of these annuli in S^3 is homeomorphic to the knot complement

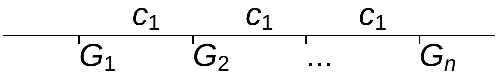
Polytopes of $S^3(A(K_i))$

Polytopes of $S^3(A(K_i))$

- $H_1(S^3(A(K_i))) \cong \mathbb{Z}$

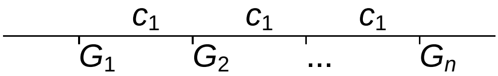
Polytopes of $S^3(A(K_i))$

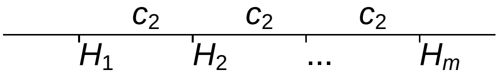
- $H_1(S^3(A(K_i))) \cong \mathbb{Z}$



Polytopes of $S^3(A(K_i))$

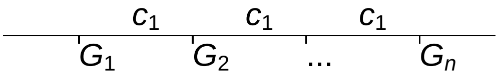
- $H_1(S^3(A(K_i))) \cong \mathbb{Z}$

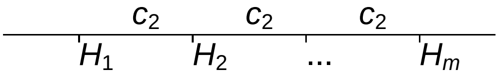




Polytopes of $S^3(A(K_i))$

- $H_1(S^3(A(K_i))) \cong \mathbb{Z}$

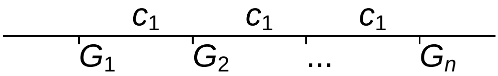


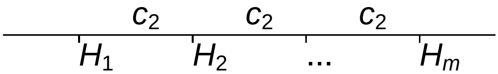


- Polytopes for $S^3(A(K_1))$ and $S^3(A(K_2))$

Polytopes of $S^3(A(K_i))$

- $H_1(S^3(A(K_i))) \cong \mathbb{Z}$





- Polytopes for $S^3(A(K_1))$ and $S^3(A(K_2))$
- G_1, G_n, H_1 and H_m are all non-zero

Tensor product formula

Tensor product formula

- We now plumb the annuli

Tensor product formula

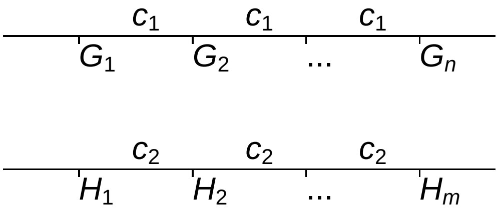
- We now plumb the annuli
- (Juhász-Ni)If a surface R is a Murasugi sum of two subsurfaces R_1 and R_2

$$SFH\left(S^3(R)\right) \cong SFH\left(S^3(R_1)\right) \otimes SFH\left(S^3(R_2)\right)$$

Polytope of $S^3(R)$

$$
\begin{array}{ccccccc}
G_1 \otimes H_1 & \overset{c_1}{\text{———}} & G_2 \otimes H_1 & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_1 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_2 & \overset{c_1}{\text{———}} & G_2 \otimes H_2 & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_2 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
\vdots & & \vdots & & & & \vdots \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_m & \overset{c_1}{\text{———}} & G_2 \otimes H_m & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_m
\end{array}
$$

Polytopes of $S^3(A(K_i))$



Polytopes for $S^3(A(K_1))$ and $S^3(A(K_2))$

Polytope of $S^3(R)$

$$
\begin{array}{ccccccccc}
G_1 \otimes H_1 & \overset{c_1}{\text{———}} & G_2 \otimes H_1 & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_1 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_2 & \overset{c_1}{\text{———}} & G_2 \otimes H_2 & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_2 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
\vdots & & \vdots & & & & \vdots \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_m & \overset{c_1}{\text{———}} & G_2 \otimes H_m & \overset{c_1}{\text{———}} & \cdots & \overset{c_1}{\text{———}} & G_n \otimes H_m
\end{array}
$$

Summarizing the results

Summarizing the results

- The polytopes of $SFH(S^3(R))$ and $SFH(S^3(R'))$ are rectangular

Summarizing the results

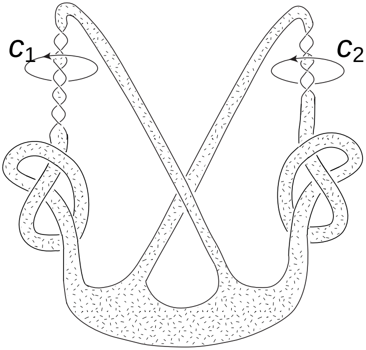
- The polytopes of $SFH(S^3(R))$ and $SFH(S^3(R'))$ are rectangular
- The corners in each rectangle have non-zero groups

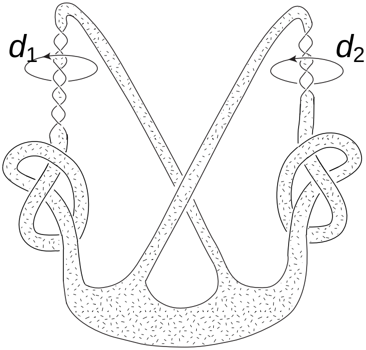
Summarizing the results

- The polytopes of $SFH(S^3(R))$ and $SFH(S^3(R'))$ are rectangular
- The corners in each rectangle have non-zero groups
- In $SFH(S^3(R))$ for instance, $G_1 \otimes H_1$, $G_1 \otimes H_m$, $G_n \otimes H_1$ and $G_n \otimes H_m$ are all non-zero

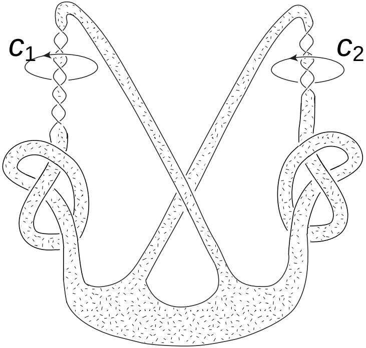
Plan of the proof

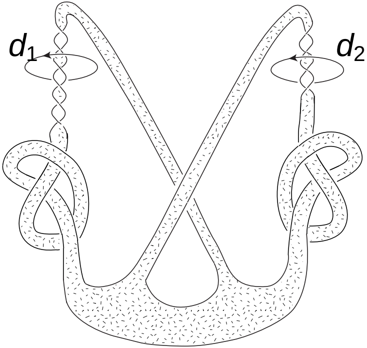
Plan of the proof





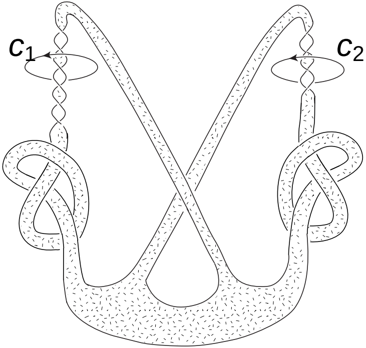
Plan of the proof

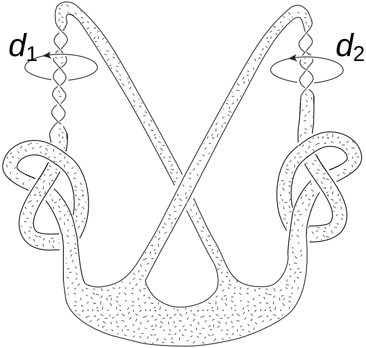




It remains to prove R and R' are inequivalent.

Plan of the proof





It remains to prove R and $R^{'}$ are inequivalent. We use a contrapositive argument.

”Proof” of the main theorem

If R and $R^{'}$ were equivalent

"Proof" of the main theorem

If R and R' were equivalent

- A function $f : S^3(R) \to S^3(R')$

"Proof" of the main theorem

If R and $R^{'}$ were equivalent

- A function $f : S^3(R) \to S^3(R^{'})$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R^{'}))$ preserves the Seifert form

"Proof" of the main theorem

If R and $R^{'}$ were equivalent

- A function $f : S^3(R) \to S^3(R^{'})$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R^{'}))$ preserves the Seifert form
  - i.e., $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3(R))$

"Proof" of the main theorem

If R and $R^{'}$ were equivalent

- A function $f : S^3(R) \to S^3(R^{'})$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R^{'}))$ preserves the Seifert form
  - i.e., $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3(R))$
- An isomorphism $\sigma : SFH(S^3(R)) \to SFH(S^3(R^{'}))$;

”Proof” of the main theorem

If R and $R^{'}$ were equivalent

- A function $f : S^3(R) \to S^3(R^{'})$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R^{'}))$ preserves the Seifert form
  - i.e., $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3(R))$
- An isomorphism $\sigma : SFH(S^3(R)) \to SFH(S^3(R^{'}))$; compatible with taking difference classes

"Proof" of the main theorem

If R and $R^{'}$ were equivalent

- A function $f : S^3(R) \to S^3(R^{'})$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R^{'}))$ preserves the Seifert form
  - i.e., $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3(R))$
- An isomorphism $\sigma : SFH(S^3(R)) \to SFH(S^3(R^{'}))$; compatible with taking difference classes
  - i.e., for $x, y, z, w \in SFH(S^3(R))$

"Proof" of the main theorem

If R and R' were equivalent

- A function $f : S^3(R) \to S^3(R')$
- $f_* : H_1(S^3(R)) \to H_1(S^3(R'))$ preserves the Seifert form
  - i.e., $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3(R))$
- An isomorphism $\sigma : SFH(S^3(R)) \to SFH(S^3(R'))$; compatible with taking difference classes
  - i.e., for $x, y, z, w \in SFH(S^3(R))$
    $\epsilon(\sigma(x), \sigma(y)).\epsilon(\sigma(z), \sigma(w)) = f_*\epsilon(x, y).f_*\epsilon(z, w)$

”Proof” of the main theorem

"Proof" of the main theorem

Take $x_{ij} \in G_i \otimes H_j$

$$
\begin{array}{ccccccc}
G_1 \otimes H_1 & \overset{c_1}{\text{———}} & G_2 \otimes H_1 & \overset{c_1}{\text{———}} & \dots & \overset{c_1}{\text{———}} & G_n \otimes H_1 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_2 & \overset{c_1}{\text{———}} & G_2 \otimes H_2 & \overset{c_1}{\text{———}} & \dots & \overset{c_1}{\text{———}} & G_n \otimes H_2 \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
\vdots & & \vdots & & & & \vdots \\
c_2 \Big| & & c_2 \Big| & & & & c_2 \Big| \\
G_1 \otimes H_m & \overset{c_1}{\text{———}} & G_2 \otimes H_m & \overset{c_1}{\text{———}} & \dots & \overset{c_1}{\text{———}} & G_n \otimes H_m
\end{array}
$$

”Proof” of the main theorem

”Proof” of the main theorem

Suppose R and R' were equivalent.

”Proof” of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction.

”Proof” of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction. For on the one hand

"Proof" of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction. For on the one hand

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nkl + mk)$$

"Proof" of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction. For on the one hand

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nkl + mk)$$

- On the other hand

"Proof" of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction. For on the one hand

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nkl + mk)$$

- On the other hand

$$\epsilon(\sigma(x_{11}), \sigma(x_{nm})).\epsilon(\sigma(x_{11}), \sigma(x_{k1})) \\ = f_*(\epsilon(x_{11}, x_{nm})).f_*(\epsilon(x_{11}, x_{k1})) = \pm nk$$

"Proof" of the main theorem

Suppose R and $R^{'}$ were equivalent.

- We get a contradiction. For on the one hand

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nkl + mk)$$

- On the other hand

$$\begin{aligned}\epsilon(\sigma(x_{11}), \sigma(x_{nm})).\epsilon(\sigma(x_{11}), \sigma(x_{k1}))\\ = f_*(\epsilon(x_{11}, x_{nm})).f_*(\epsilon(x_{11}, x_{k1})) = \pm nk\end{aligned}$$

- Therefore, $R \not\approx R^{'}$.

Thank you!