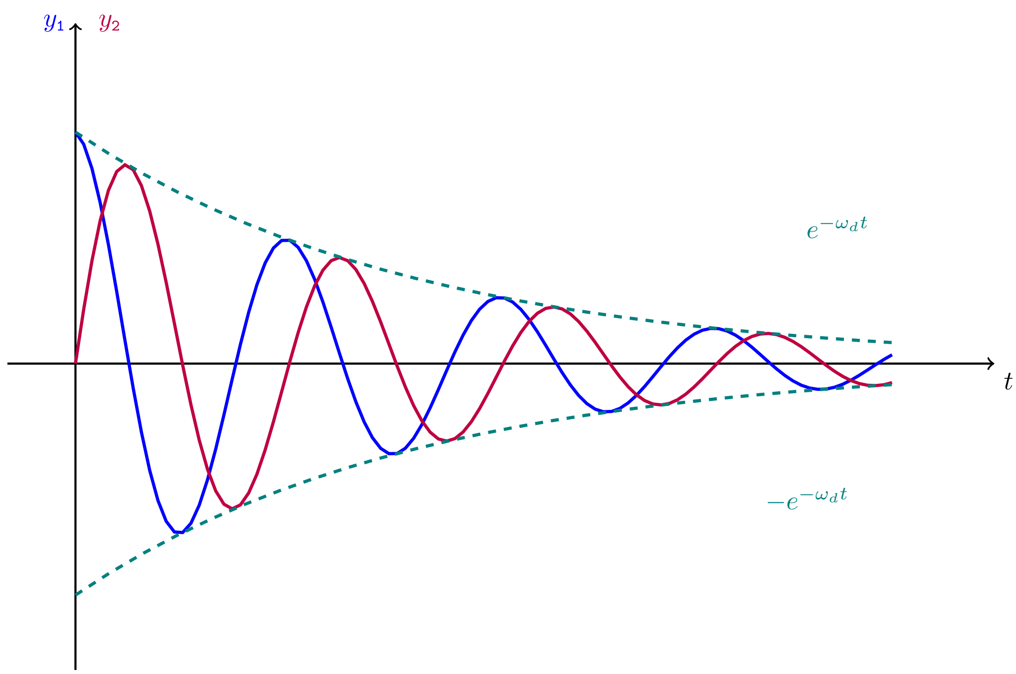
Chapter 3. Second Order Linear Equations

Newton's second law of motion, $ma = f$, is maybe one of the first differential equations written. This is a second order equation, since the acceleration is the second time derivative of the particle position function. Second order differential equations are more difficult to solve than first order equations. In § 3.1 we compare results on linear first and second order equations. While there is an explicit formula for all solutions to first order linear equations, not such formula exists for all solutions to second order linear equations. The most one can get is the result in Theorem 3.1.7. In the following couple of sections we find explicit formulas for all solutions to linear second order equations that are both homogeneous and with constant coefficients. These formulas are then generalized to nonhomogeneous equations.



3.1. General Second Order Linear Equations. Superposition Property

We studied first order linear equations in § **??**-**??**, where we obtained a formula for all solutions to these equations. We could say that we know all that can be known about solutions to *first order linear* equations. However, this is not the case for solutions to second order linear equations, since we do not have a general formula for all solutions to these equations.

In this section we present two main results, the first one is Theorem 3.1.2, which says that there are solutions to second order linear equations when the equation coefficients are continuous functions. Furthermore, these solutions have two free parameters that can be fixed by appropriate initial conditions.

The second result is Theorem 3.1.7, which is the closest we can get to a formula for solutions to second order linear equations without sources—homogeneous equations. To know all solutions to these equations we only need to know two solutions that are not proportional to each other. The proof of Theorem 3.1.7 is based on Theorem 3.1.2 plus an algebraic calculation and properties of the Wronskian function, which are derived from Abel's Theorem.

3.1.1. Definitions and Examples.

We start with a definition of second order linear differential equations. After a few examples we state the first of the main results, Theorem 3.1.2, about existence and uniqueness of solutions to an initial value problem in the case that the equation coefficients are continuous functions.

Definition 3.1.1. *A **second order linear differential equation** for the function y is*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \tag{3.1.1}$$

where a_1, a_0, b are given functions on the interval $I \subset \mathbb{R}$. The Eq. (3.1.1) above:

*(a) is **homogeneous** iff the source $b(t) = 0$ for all $t \in \mathbb{R}$;*
*(b) has **constant coefficients** iff a_1 and a_0 are constants;*
*(c) has **variable coefficients** iff either a_1 or a_0 is not constant.*

Remark: The notion of an homogeneous equation presented here is different from the Euler homogeneous equations we studied in § **??**.

Example 3.1.1:

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

(b) A second order, linear, nonhomogeneous, constant coefficients, equation is

$$y'' - 3y' + y = \cos(3t).$$

(c) A second order, linear, nonhomogeneous, variable coefficients equation is

$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's law of motion for a point particle of mass m moving in one space dimension under a force f is mass times acceleration equals force,

$$m\, y''(t) = f(t, y(t), y'(t)).$$

(e) Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

$$-\frac{\hbar^2}{2m}\, \psi'' + V(x)\, \psi = E\, \psi,$$

where ψ is the probability density of finding a particle of mass m at the position x having energy E under a potential V, where $\hbar$ is Planck constant divided by 2π. ◁

Example 3.1.2: Find the differential equation satisfied by the family of functions

$$y(t) = c_1 e^{4t} + c_2 e^{-4t},$$

where c_1, c_2 are arbitrary constants.

Solution: From the definition of y compute c_1,

$$c_1 = y e^{-4t} - c_2 e^{-8t}.$$

Now compute the derivative of function y

$$y' = 4c_1 e^{4t} - 4c_2 e^{-4t},$$

Replace c_1 from the first equation above into the expression for y',

$$y' = 4(y e^{-4t} - c_2 e^{-8t})e^{4t} - 4c_2 e^{-4t} \quad \Rightarrow \quad y' = 4y + (-4-4)c_2 e^{-4t},$$

so we get an expression for c_2 in terms of y and y',

$$y' = 4y - 8c_2 e^{-4t} \quad \Rightarrow \quad c_2 = \frac{1}{8}(4y - y')\, e^{4t}$$

At this point we can compute c_1 in terms of y and y', although we do not need it for what follows. Anyway,

$$c_1 = y e^{-4t} - \frac{1}{8}(4y - y')e^{4t}e^{-8t} \quad \Rightarrow \quad c_1 = \frac{1}{8}\,(4y + y')\, e^{-4t}.$$

We do not need c_1 because we can get a differential equation for y from the equation for c_2. Compute the derivative of that equation,

$$0 = c_2' = \frac{1}{2}(4y - y')\, e^{4t} + \frac{1}{8}(4y' - y'')\, e^{4t} \quad \Rightarrow \quad 4(4y - y') + (4y' - y'') = 0$$

which gives us the following second order linear differential equation for y,

$$y'' - 16\, y = 0.$$

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Example 3.1.3: Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2\, t, \qquad c_1, c_2 \in \mathbb{R}.$$

Solution: Compute $y' = -\dfrac{c_1}{t^2} + c_2$. Get one constant from y' and put it in y,

$$c_2 = y' + \frac{c_1}{t^2} \quad \Rightarrow \quad y = \frac{c_1}{t} + \Big(y' + \frac{c_1}{t^2}\Big)\, t,$$

so we get

$$y = \frac{c_1}{t} + t\, y' + \frac{c_1}{t} \quad \Rightarrow \quad y = \frac{2c_1}{t} + t\, y'.$$

Compute the constant from the expression above,

$$\frac{2c_1}{t} = y - t\, y' \quad \Rightarrow \quad 2c_1 = t\, y - t^2\, y'.$$

Since the left hand side is constant,

$$0 = (2c_1)' = (t\, y - t^2\, y')' = y + t\, y' - 2t\, y' - t^2\, y'',$$

so we get that y must satisfy the differential equation

$$t^2\, y'' + t\, y' - y = 0.$$

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EXAMPLE 3.1.4: Find the differential equation satisfied by the family of functions

$$y(x) = c_1\, x + c_2\, x^2,$$

where c_1, c_2 are arbitrary constants.

SOLUTION: Compute the derivative of function y

$$y'(x) = c_1 + 2c_2\, x,$$

From here it is simple to get c_1,

$$c_1 = y' - 2c_2\, x.$$

Use this expression for c_1 in the expression for y,

$$y = (y' - 2c_2\, x)\, x + c_2\, x^2 = x\, y' - c_2\, x^2 \quad \Rightarrow \quad c_2 = \frac{y'}{x} - \frac{y}{x^2}.$$

To get the differential equation for y we do not need c_1, but we compute it anyway,

$$c_1 = y' - 2(\frac{y'}{x} - \frac{y}{x^2})\, x = y' - 2\, y' + \frac{2y}{x} \quad \Rightarrow \quad c_1 = -y' + \frac{2y}{x}.$$

The equation for y can be obtained computing a derivative in the expression for c_2,

$$0 = c_2' = \frac{y''}{x} - \frac{y'}{x^2} - \frac{y'}{x^2} + 2\frac{y}{x^3} = \frac{y''}{x} - 2\frac{y'}{x^2} + 2\frac{y}{x^3} = 0 \quad \Rightarrow \quad x^2\, y'' - 2x\, y' + 2\, y = 0.$$

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3.1.2. **Solutions to the Initial Value Problem.** Here is the first of the two main results in this section. Second order linear differential equations have solutions in the case that the equation coefficients are continuous functions. Since the solution is unique when we specify two initial conditions, the general solution must have two arbitrary integration constants.

Theorem 3.1.2 (IVP)**.** *If the functions a_1, a_0, b are continuous on a closed interval $I \subset \mathbb{R}$, the constant $t_0 \in I$, and y_0, $y_1 \in \mathbb{R}$ are arbitrary constants, then there is a unique solution y, defined on I, of the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{3.1.2}$$

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem **??** can be extended to prove Theorem 3.1.2.

EXAMPLE 3.1.5: Find the domain of the solution to the initial value problem

$$(t-1)\, y'' - 3t\, y' + \frac{4(t-1)}{(t-3)}\, y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

SOLUTION: We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{(t-1)}\, y' + \frac{4}{(t-3)}\, y = t.$$

The equation coefficients are defined on the domain

$$(-\infty, 1) \cup (1, 3) \cup (3, \infty).$$

So the solution may not be defined at $t = 1$ or $t = 3$. That is, the solution is defined in

$$(-\infty, 1) \qquad \text{or} \qquad (1, 3) \qquad \text{or} \qquad (3, \infty).$$

Since the initial condition is at $t_0 = 2 \in (1, 3)$, then the domain of the solution is

$$D = (1, 3).$$

◁

3.1.3. **Properties of Homogeneous Equations.** We simplify the problem with the hope to get deeper properties of its solutions. From now on in this section we focus on homogeneous equations only. We will get back to non-homogeneous equations in a later section. But before getting into homogeneous equations, we introduce a new notation to write differential equations. This is a shorter, more economical, notation. Given two functions a_1, a_0, introduce the function L acting on a function y, as follows,

$$L(y) = y'' + a_1(t)\, y' + a_0(t)\, y. \tag{3.1.3}$$

The function L acts on the function y and the result is another function, given by Eq. (3.1.3).

Example 3.1.6: Compute the operator $L(y) = t\, y'' + 2y' - \frac{8}{t}\, y$ acting on $y(t) = t^3$.

Solution: Since $y(t) = t^3$, then $y'(t) = 3t^2$ and $y''(t) = 6t$, hence

$$L(t^3) = t\,(6t) + 2(3t^2) - \frac{8}{t}\, t^3 \quad \Rightarrow \quad L(t^3) = 4t^2.$$

The function L acts on the function $y(t) = t^3$ and the result is the function $L(t^3) = 4t^2$. ◁

The function L above is called *an operator*, to emphasize that L is a function that acts on other functions, instead of acting on numbers, as the functions we are used to. The operator L above is also called a *differential operator*, since $L(y)$ contains derivatives of y. These operators are useful to write differential equations in a compact notation, since

$$y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$$

can be written using the operator $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ as

$$L(y) = f.$$

An important type of operators are the linear operators.

Definition 3.1.3. *An operator L is a* ***linear operator*** *iff for every pair of functions y_1, y_2 and constants c_1, c_2 holds*

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2). \tag{3.1.4}$$

In this Section we work with linear operators, as the following result shows.

Theorem 3.1.4 (Linear Operator). *The operator $L(y) = y'' + a_1\, y' + a_0\, y$, where a_1, a_0 are continuous functions and y is a twice differentiable function, is a linear operator.*

Proof of Theorem 3.1.4: This is a straightforward calculation:

$$L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)'' + a_1\,(c_1 y_1 + c_2 y_2)' + a_0\,(c_1 y_1 + c_2 y_2).$$

Recall that derivations is a linear operation and then reoorder terms in the following way,

$$L(c_1 y_1 + c_2 y_2) = \big(c_1 y_1'' + a_1\, c_1 y_1' + a_0\, c_1 y_1\big) + \big(c_2 y_2'' + a_1\, c_2 y_2' + a_0\, c_2 y_2\big).$$

Introduce the definition of L back on the right-hand side. We then conclude that

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

This establishes the Theorem. □

The linearity of an operator L translates into the superposition property of the solutions to the homogeneous equation $L(y) = 0$.

Theorem 3.1.5 (Superposition)**.** *If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations $L(y_1) = 0$, $L(y_2) = 0$, then for every constants c_1, c_2 holds*

$$L(c_1\, y_1 + c_2\, y_2) = 0.$$

Remark: This result is *not true* for nonhomogeneous equations.

Proof of Theorem 3.1.5: Verify that the function $y = c_1 y_1 + c_2 y_2$ satisfies $L(y) = 0$ for every constants c_1, c_2, that is,

$$L(y) = L(c_1 y_1 + c_2 y_2) = c_1\, L(y_1) + c_2\, L(y_2) = c_1\, 0 + c_2\, 0 = 0.$$

This establishes the Theorem. □

We now introduce the notion of linearly dependent and linearly independent functions.

Definition 3.1.6. *Two functions y_1, y_2 are called **linearly dependent** iff they are proportional. Otherwise, the functions are **linearly independent**.*

Remarks:

(a) Two functions y_1, y_2 are proportional iff there is a constant c such that for all t holds

$$y_1(t) = c\, y_2(t).$$

(b) The function $y_1 = 0$ is proportional to every other function y_2, since holds $y_1 = 0 = 0\, y_2$.

The definitions of linearly dependent or independent functions found in the literature are equivalent to the definition given here, but they are worded in a slight different way. Often in the literature, two functions are called linearly dependent on the interval I iff there exist constants c_1, c_2, not both zero, such that for all $t \in I$ holds

$$c_1 y_1(t) + c_2 y_2(t) = 0.$$

Two functions are called linearly independent on the interval I iff they are not linearly dependent, that is, the only constants c_1 and c_2 that for all $t \in I$ satisfy the equation

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

are the constants $c_1 = c_2 = 0$. This wording makes it simple to generalize these definitions to an arbitrary number of functions.

Example 3.1.7:

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are linearly dependent.
(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are linearly independent.

Solution:

Part (a): This is trivial, since $2y_1(t) - y_2(t) = 0$.

Part (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2}\, c_2 = 0, \quad c_1 + \frac{3\pi}{2}\, c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are linearly independent. ◁

We now introduce the second main result in this section. If you know two linearly independent solutions to a second order linear homogeneous differential equation, then you actually know all possible solutions to that equation. Any other solution is just a linear combination of the previous two solutions. We repeat that the equation must be homogeneous. This is the closer we can get to a general formula for solutions to second order linear homogeneous differential equations.

Theorem 3.1.7 (General Solution)**.** *If y_1 and y_2 are linearly independent solutions of the equation $L(y) = 0$ on an interval $I \subset \mathbb{R}$, where $L(y) = y'' + a_1\, y' + a_0\, y$, and a_1, a_2 are continuous functions on I, then there are unique constants c_1, c_2 such that every solution y of the differential equation $L(y) = 0$ on I can be written as a linear combination*

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

Before we prove Theorem 3.1.7, it is convenient to state the following the definitions, which come out naturally from this Theorem.

Definition 3.1.8.

*(a) The functions y_1 and y_2 are **fundamental solutions** of the equation $L(y) = 0$ iff y_1, y_2 are linearly independent and*

$$L(y_1) = 0, \qquad L(y_2) = 0.$$

*(b) The **general solution** of the homogeneous equation $L(y) = 0$ is a two-parameter family of functions $y_{\rm gen}$ given by*

$$y_{\rm gen}(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

where the arbitrary constants c_1, c_2 are the parameters of the family, and y_1, y_2 are fundamental solutions of $L(y) = 0$.

Example 3.1.8: Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

Solution: We first show that y_1 and y_2 are solutions to the differential equation, since

$$L(y_1) = y_1'' + y_1' - 2y_1 = e^t + e^t - 2e^t = (1 + 1 - 2)e^t = 0,$$

$$L(y_2) = y_2'' + y_2' - 2y_2 = 4\, e^{-2t} - 2\, e^{-2t} - 2e^{-2t} = (4 - 2 - 2)e^{-2t} = 0.$$

It is not difficult to see that y_1 and y_2 are linearly independent. It is clear that they are not proportional to each other. A proof of that statement is the following: Find the constants c_1 and c_2 such that

$$0 = c_1\, y_1 + c_2\, y_2 = c_1\, e^t + c_2\, e^{-2t} \quad t \in \mathbb{R} \quad \Rightarrow \quad 0 = c_1\, e^t - 2c_2\, e^{-2t}$$

The second equation is the derivative of the first one. Take $t = 0$ in both equations,

$$0 = c_1 + c_2, \quad 0 = c_1 - 2c_2 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

We conclude that y_1 and y_2 are fundamental solutions to the differential equation above.$\lhd$

Remark: The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

$$y_1(t) = \frac{2}{3} e^t + \frac{1}{3} e^{-2t}, \qquad y_2(t) = \frac{1}{3}\big(e^t - e^{-2t}\big).$$

To prove Theorem 3.1.7 we need to introduce the Wronskian function and to verify some of its properties. The Wronskian function is studied in the following Subsection and Abel's

Theorem is proved. Once that is done we can say that the proof of Theorem 3.1.7 is complete.

Proof of Theorem 3.1.7: We need to show that, given any fundamental solution pair, y_1, y_2, any other solution y to the homogeneous equation $L(y) = 0$ must be a unique linear combination of the fundamental solutions,

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t), \tag{3.1.5}$$

for appropriately chosen constants c_1, c_2.

First, the superposition property implies that the function y above is solution of the homogeneous equation $L(y) = 0$ for every pair of constants c_1, c_2.

Second, given a function y, if there exist constants c_1, c_2 such that Eq. (3.1.5) holds, then these constants are unique. The reason is that functions y_1, y_2 are linearly independent. This can be seen from the following argument. If there are another constants $\tilde{c}_1$, $\tilde{c}_2$ so that

$$y(t) = \tilde{c}_1\, y_1(t) + \tilde{c}_2\, y_2(t),$$

then subtract the expression above from Eq. (3.1.5),

$$0 = (c_1 - \tilde{c}_1)\, y_1 + (c_2 - \tilde{c}_2)\, y_2 \quad \Rightarrow \quad c_1 - \tilde{c}_1 = 0, \quad c_2 - \tilde{c}_2 = 0,$$

where we used that y_1, y_2 are linearly independent. This second part of the proof can be obtained from the part three below, but I think it is better to highlight it here.

So we only need to show that the expression in Eq. (3.1.5) contains all solutions. We need to show that we are not missing any other solution. In this third part of the argument enters Theorem 3.1.2. This Theorem says that, in the case of homogeneous equations, the initial value problem

$$L(y) = 0, \qquad y(t_0) = d_1, \quad y'(t_0) = d_2,$$

always has a unique solution. That means, a good parametrization of all solutions to the differential equation $L(y) = 0$ is given by the two constants, d_1, d_2 in the initial condition. To finish the proof of Theorem 3.1.7 we need to show that the constants c_1 and c_2 are also good to parametrize all solutions to the equation $L(y) = 0$. One way to show this, is to find an invertible map from the constants d_1, d_2, which we know parametrize all solutions, to the constants c_1, c_2. The map itself is simple to find,

$$\begin{aligned} d_1 &= c_1\, y_1(t_0) + c_2\, y_2(t_0) \\ d_2 &= c_1\, y_1'(t_0) + c_2\, y_2'(t_0). \end{aligned}$$

We now need to show that this map is invertible. From linear algebra we know that this map acting on c_1, c_2 is invertible iff the determinant of the coefficient matrix is nonzero,

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0.$$

This leads us to investigate the function

$$W_{12}(t) = y_1(t)\, y_2'(t) - y_1'(t) y_2(t)$$

This function is called the Wronskian of the two functions y_1, y_2. At the end of this section we prove Theorem 3.1.13, which says the following: If y_1, y_2 are fundamental solutions of $L(y) = 0$ on $I \subset \mathbb{R}$, then $W_{12}(t) \neq 0$ on I. This statement establishes the Theorem. $\square$

3.1.4. **The Wronskian Function.** We now introduce a function that provides important information about the linear dependency of two functions y_1, y_2. This function, W, is called the Wronskian to honor the polish scientist Josef Wronski, who first introduced this function in 1821 while studying a different problem.

Definition 3.1.9. *The* ***Wronskian*** *of the differentiable functions* y_1, y_2 *is the function*

$$W_{12}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Remark: Introducing the matrix valued function $A(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$ the Wronskian can be written using the determinant of that 2×2 matrix, $W_{12}(t) = \det\big(A(t)\big)$. An alternative notation is: $W_{12} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

EXAMPLE 3.1.9: Find the Wronskian of the functions:
(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)
(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

SOLUTION:
Part (a): By the definition of the Wronskian:

$$W_{12}(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix} = \sin(t)2\cos(t) - \cos(t)2\sin(t)$$

We conclude that $W_{12}(t) = 0$. Notice that y_1 and y_2 are linearly dependent.

Part (b): Again, by the definition of the Wronskian:

$$W_{12}(t) = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix} = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t)t\sin(t).$$

We conclude that $W_{12}(t) = \sin^2(t)$. Notice that y_1 and y_2 are linearly independent. $\lhd$

It is simple to prove the following relation between the Wronskian of two functions and the linear dependency of these two functions.

Theorem 3.1.10 (Wronskian I)**.** *If* y_1, y_2 *are linearly dependent on* $I \subset \mathbb{R}$, *then*

$$W_{12} = 0 \quad on \quad I.$$

Proof of Theorem 3.1.10: Since the functions y_1, y_2 are linearly dependent, there exists a nonzero constant c such that $y_1 = c\,y_2$; hence holds,

$$W_{12} = y_1\,y_2' - y_1'\,y_2 = (c\,y_2)\,y_2' - (c\,y_2)'\,y_2 = 0.$$

This establishes the Theorem. □

Remark: The converse statement to Theorem 3.1.10 is false. If $W_{12}(t) = 0$ for all $t \in I$, that ***does not*** imply that y_1 and y_2 are linearly dependent.

EXAMPLE 3.1.10: Show that the functions

$$y_1(t) = t^2, \quad \text{and} \quad y_2(t) = |t|\,t, \quad \text{for} \quad t \in \mathbb{R}$$

are linearly independent and have Wronskian $W_{12} = 0$.

SOLUTION:
First, these functions are linearly independent, since $y_1(t) = -y_2(t)$ for $t < 0$, but $y_1(t) = y_2(t)$ for $t > 0$. So there is not c such that $y_1(t) = c\,y_2(t)$ for all $t \in \mathbb{R}$.

Second, their Wronskian vanishes on $\mathbb{R}$. This is simple to see, since $y_1(t) = -y_2(t)$ for $t < 0$, then $W_{12} = 0$ for $t < 0$. Since $y_1(t) = y_2(t)$ for $t > 0$, then $W_{12} = 0$ for $t > 0$. Finally, it is not difficult to see that $W_{12}(t = 0) = 0$. $\lhd$

Remark: Often in the literature one finds the negative of Theorem 3.1.10, which is equivalent to Theorem 3.1.10, and we summarize ibn the followng Corollary.

Corollary 3.1.11 (Wronskian I). *If the Wronskian $W_{12}(t_0) \neq 0$ at a point $t_0 \in I$, then the functions y_1, y_2 defined on I are linearly independent.*

The results mentioned above provide different properties of the Wronskian of two functions. But none of these results is what we need to finish the proof of Theorem 3.1.7. In order to finish that proof we need one more result, Abel's Theorem.

3.1.5. **Abel's Theorem.** We now show that the Wronskian of two solutions of a differential equation satisfies a differential equation of its own. This result is known as Abel's Theorem.

Theorem 3.1.12 (Abel). *If y_1, y_2 are twice continuously differentiable solutions of*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{3.1.6}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian W_{12} satisfies

$$W_{12}' + a_1(t)\, W_{12} = 0.$$

Therefore, for any $t_0 \in I$, the Wronskian W_{12} is given by the expression

$$W_{12}(t) = W_{12}(t_0)\, e^{-A_1(t)},$$

where $A_1(t) = \int_{t_0}^{t} a_1(s)\, ds$.

Proof of Theorem 3.1.12: We start computing the derivative of the Wronskian function,

$$W_{12}' = \big(y_1\, y_2' - y_1'\, y_2\big)' = y_1\, y_2'' - y_1''\, y_2.$$

Recall that both y_1 and y_2 are solutions to Eq. (3.1.6), meaning,

$$y_1'' = -a_1\, y_1' - a_0\, y_1, \qquad y_2'' = -a_1\, y_2' - a_0\, y_2.$$

Replace these expressions in the formula for W_{12}' above,

$$W_{12}' = y_1\,\big(-a_1\, y_2' - a_0\, y_2\big) - \big(-a_1\, y_1' - a_0\, y_1\big)\, y_2 \quad \Rightarrow \quad W_{12}' = -a_1\big(y_1\, y_2' - y_1'\, y_2\big)$$

So we obtain the equation

$$W_{12}' + a_1(t)\; W_{12} = 0.$$

This equation for W_{12} is a first order linear equation; its solution can be found using the method of integrating factors, given in Section **??**, which results is the expression in the Theorem 3.1.12. This establishes the Theorem. $\square$

We now show one application of Abel's Theorem.

Example 3.1.11: Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Notice that we do not known the explicit expression for the solutions. Nevertheless, Theorem 3.1.12 says that we can compute their Wronskian. First, we have to rewrite the differential equation in the form given in that Theorem, namely,

$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

Then, Theorem 3.1.12 says that the Wronskian satisfies the differential equation

$$W_{12}'(t) - \left(\frac{2}{t} + 1\right) W_{12}(t) = 0.$$

This is a first order, linear equation for W_{12}, so its solution can be computed using the method of integrating factors. That is, first compute the integral

$$\begin{aligned} -\int_{t_0}^{t} \left(\frac{2}{s} + 1\right) ds &= -2\ln\left(\frac{t}{t_0}\right) - (t - t_0) \\ &= \ln\left(\frac{t_0^2}{t^2}\right) - (t - t_0). \end{aligned}$$

Then, the integrating factor μ is given by

$$\mu(t) = \frac{t_0^2}{t^2}\, e^{-(t-t_0)},$$

which satisfies the condition $\mu(t_0) = 1$. So the solution, W_{12} is given by

$$\Big(\mu(t)W_{12}(t)\Big)' = 0 \quad \Rightarrow \quad \mu(t)W_{12}(t) - \mu(t_0)W_{12}(t_0) = 0$$

so, the solution is

$$W_{12}(t) = W_{12}(t_0)\frac{t^2}{t_0^2}\, e^{(t-t_0)}.$$

If we call the constant $c = W_{12}(t_0)/[t_0^2 e^{t_0}]$, then the Wronskian has the simpler form

$$W_{12}(t) = c\, t^2 e^t.$$

$\lhd$

We now state and prove the statement we need to complete the proof of Theorem 3.1.7.

Theorem 3.1.13 (Wronskian II)**.** *If y_1, y_2 are fundamental solutions of $L(y) = 0$ on $I \subset \mathbb{R}$, then $W_{12}(t) \neq 0$ on I.*

Remark: Instead of proving the Theorem above, we prove an equivalent statement—the negative statement.

Corollary 3.1.14 (Wronskian II)**.** *If y_1, y_2 are solutions of $L(y) = 0$ on $I \subset \mathbb{R}$ and there is a point $t_1 \in I$ such that $W_{12}(t_1) = 0$, then y_1, y_2 are linearly dependent on I.*

Proof of Corollary 3.1.14: We know that y_1, y_2 are solutions of $L(y) = 0$. Then, Abel's Theorem says that their Wronskian W_{12} is given by

$$W_{12}(t) = W_{12}(t_0)\, e^{-A_1(t)},$$

for any $t_0 \in I$. Chossing the point t_0 to be t_1, the point where by hypothesis $W_{12}(t_1) = 0$, we get that

$$W_{12}(t) = 0 \quad \text{for all} \quad t \in I.$$

Knowing that the Wronskian vanishes identically on I, we can write

$$y_1\, y_2' - y_1'\, y_2 = 0,$$

on I. If either y_1 or y_2 is the function zero, then the set is linearly dependent. So we can assume that both are not identically zero. Let's assume there exists $t_1 \in I$ such that

$y_1(t_1) \neq 0$. By continuity, y_1 is nonzero in an open neighborhood $I_1 \subset I$ of t_1. So in that neighborhood we can divide the equation above by y_1^2,

$$\frac{y_1\, y_2' - y_1'\, y_2}{y_1^2} = 0 \quad \Rightarrow \quad \Big(\frac{y_2}{y_1}\Big)' = 0 \quad \Rightarrow \quad \frac{y_2}{y_1} = c, \qquad \text{on } \ I_1,$$

where $c \in \mathbb{R}$ is an arbitrary constant. So we conclude that y_1 is proportional to y_2 on the open set I_1. That means that the function $y(t) = y_2(t) - c\, y_1(t)$, satisfies

$$L(y) = 0, \qquad y(t_1) = 0, \quad y'(t_1) = 0.$$

Therefore, the existence and uniqueness Theorem 3.1.2 says that $y(t) = 0$ for all $t \in I$. This finally shows that y_1 and y_2 are linearly dependent. This establishes the Theorem. $\square$

3.1.6. **Exercises.**

3.1.1.- Compute the Wronskian of the following functions:

(a) $f(t) = \sin(t)$, $g(t) = \cos(t)$.
(b) $f(x) = x$, $g(x) = x\, e^x$.
(c) $f(\theta) = \cos^2(\theta)$, $g(\theta) = 1 + \cos(2\theta)$.

3.1.2.- Find the longest interval where the solution y of the initial value problems below is defined. (Do not try to solve the differential equations.)

(a) $t^2 y'' + 6y = 2t$, $y(1) = 2$, $y'(1) = 3$.
(b) $(t-6)y' + 3ty' - y = 1$, $y(3) = -1$, $y'(3) = 2$.

3.1.3.- (a) Verify that $y_1(t) = t^2$ and $y_2(t) = 1/t$ are solutions to the differential equation

$$t^2 y'' - 2y = 0, \qquad t > 0.$$

(b) Show that $y(t) = a\, t^2 + \dfrac{b}{t}$ is solution of the same equation for all constants $a, b \in \mathbb{R}$.

3.1.4.- If the graph of y, solution to a second order linear differential equation $L(y(t)) = 0$ on the interval $[a, b]$, is tangent to the t-axis at any point $t_0 \in [a, b]$, then find the solution y explicitly.

3.1.5.- (HW) Can the function $y(t) = \sin(t^2)$ be solution on an open interval containing $t = 0$ of a differential equation

$$y'' + a(t)\, y' + b(t) y = 0,$$

with continuous coefficients a and b? Explain your answer.
Hint: Assume it does satisfy an equation of this form. What equations should $a(t)$ and $b(t)$ then satisfy? Evaluate this equation at $t = 0$.

3.1.6.- Verify whether the functions y_1, y_2 below are a fundamental set for the differential equations given below:

(a) $y_1(t) = \cos(2t)$, $y_2(t) = \sin(2t)$,
$$y'' + 4y = 0.$$
(b) $y_1(t) = e^t$, $y_2(t) = t\, e^t$,
$$y'' - 2y' + y = 0.$$
(c) $y_1(x) = x$, $y_2(t) = x\, e^x$,
$$x^2\, y'' - 2x(x+2)\, y' + (x+2)\, y = 0.$$

3.1.7.- If the Wronskian of any two solutions of the differential equation

$$y'' + p(t)\, y' + q(t)\, y = 0$$

is constant, what does this imply about the coefficients p and q?

3.1.8.- Let $y(t) = c_1\, t + c_2\, t^2$ be the general solution of a second order linear differential equation $L(y) = 0$. By eliminating the constants c_1 and c_2, find the differential equation satisfied by y.