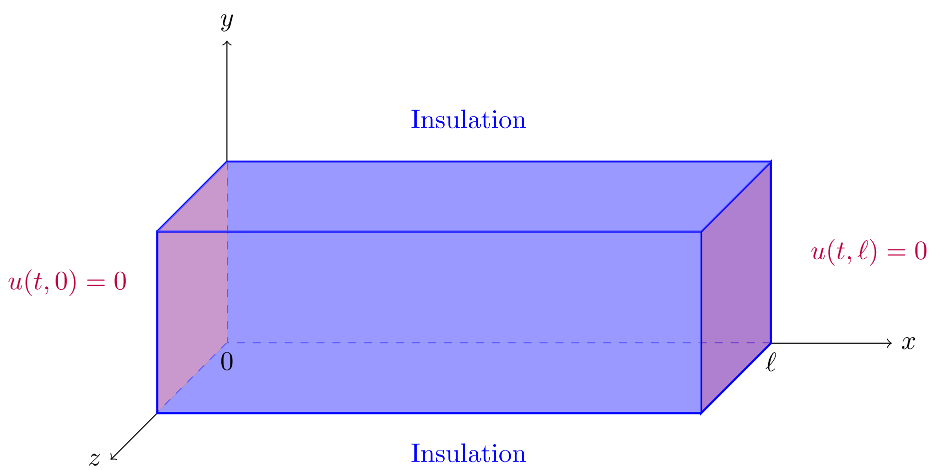
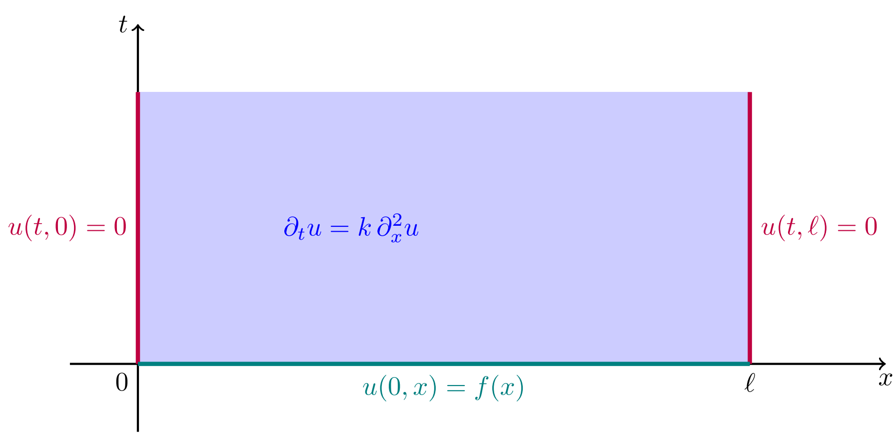
CHAPTER 9. BOUNDARY VALUE PROBLEMS

We study the a simple case of the Sturm-Liouville Problem, we then present how to compute the Fourier series expansion of continuous and discontinuous functions. We end this chapter introducing the separation of variables method to find solutions of a partial differential equation, the heat equation.





9.1. Eigenfunction Problems

In this Section we consider second order, linear, ordinary differential equations. In the first half of the Section we study boundary value problems for these equations and in the second half we focus on a particular type of boundary value problems, called the eigenvalue-eigenfunction problem for these equations.

9.1.1. **Two-Point Boundary Value Problems.**

We start with the definition of a two-point boundary value problem.

Definition 9.1.1. *A **two-point boundary value problem** (BVP) is the following: Find solutions to the differential equation*

$$y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$$

satisfying the boundary conditions (BC)

$$b_1\, y(x_1) + b_2\, y'(x_1) = y_1,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = y_2,$$

*where b_1, b_2, $\tilde{b}_1$, $\tilde{b}_2$, x_1, x_2, y_1, and y_2 are given and $x_1 \neq x_2$. The boundary conditions are **homogeneous** iff $y_1 = 0$ and $y_2 = 0$*

Remarks:

(a) The two boundary conditions are held at *different* points, $x_1 \neq x_2$.
(b) Both y and y' may appear in the boundary condition.

Example 9.1.1: We now show four examples of boundary value problems that differ only on the boundary conditions: Solve the different equation

$$y'' + a_1\, y' + a_0\, y = e^{-2t}$$

with the boundary conditions at $x_1 = 0$ and $x_2 = 1$ given below.

(a)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 1, & b_2 = 0, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 0. \end{Bmatrix}$$

(b)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 1, & b_2 = 0, \\ \tilde{b}_1 = 0, & \tilde{b}_2 = 1. \end{Bmatrix}$$

(c)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 0, & b_2 = 1, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 0. \end{Bmatrix}$$

(d)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 0, & b_2 = 1, \\ \tilde{b}_1 = 0, & \tilde{b}_2 = 1. \end{Bmatrix}$$

(e)

$$\text{BC:} \quad \begin{Bmatrix} 2\, y(0) + y'(0) = y_1, \\ y'(1) + 3\, y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 2, & b_2 = 1, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 3. \end{Bmatrix}$$

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9.1.2. **Comparison: IVP and BVP.** We now review the initial boundary value problem for the equation above, which was discussed in Sect. **??**, where we showed in Theorem **??** that this initial value problem always has a unique solution.

Definition 9.1.2 (IVP)**.** *Find all solutions of the differential equation* $y'' + a_1\, y' + a_0\, y = 0$ *satisfying the initial condition (IC)*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1. \tag{9.1.1}$$

Remarks: In an initial value problem we usually the following happens.

- The variable t represents time.
- The variable y represents position.
- The IC are position and velocity at the initial time.

A typical boundary value problem that appears in many applications is the following.

Definition 9.1.3 (BVP)**.** *Find all solutions of the differential equation* $y'' + a_1\, y' + a_0\, y = 0$ *satisfying the boundary condition (BC)*

$$y(0) = y_0, \qquad y(L) = y_1, \qquad L \neq 0. \tag{9.1.2}$$

Remarks: In a boundary value problem we usually the following happens.

- The variable x represents position.
- The variable y may represents a physical quantity such us temperature.
- The BC are the temperature at two different positions.

The names "initial value problem" and "boundary value problem" come from physics. An example of the former is to solve Newton's equations of motion for the position function of a point particle that starts at a given initial position and velocity. An example of the latter is to find the equilibrium temperature of a cylindrical bar with thermal insulation on the round surface and held at constant temperatures at the top and bottom sides.

Let's recall an important result we saw in § **??** about solutions to initial value problems.

Theorem 9.1.4 (IVP)**.** *The equation* $y'' + a_1\, y' + a_0\, y = 0$ *with IC* $y(t_0) = y_0$ *and* $y'(t_0) = y_1$ *has a unique solution* y *for each choice of the IC.*

The solutions to boundary value problems are more complicated to describe. A boundary value problem may have a unique solution, or may have infinitely many solutions, or may have no solution, depending on the boundary conditions. In the case of the boundary value problem in Def. 9.1.3 we get the following.

Theorem 9.1.5 (BVP)**.** *The equation* $y'' + a_1\, y' + a_0\, y = 0$ *with BC* $y(0) = y_0$ *and* $y(L) = y_1$, *wilt* $L \neq 0$ *and with* $r_\pm$ *roots of the characteristic polynomial* $p(r) = r^2 + a_1 r + a_0$, *satisfy the following.*

(A) If $r_+ \neq r_-$ *are reals, then the BVP above has a unique solution for all* y_0, $y_1 \in \mathbb{R}$.

(B) If $r_\pm = \alpha \pm i\,\beta$ *are complex, with* α, $\beta \in \mathbb{R}$, *then the solution of the BVP above belongs to one of the following three possibilities:*

- *(i) There exists a unique solution;*
- *(ii) There exists infinitely many solutions;*
- *(iii) There exists no solution.*

Proof of Theorem 9.1.5:
Part (A): If $r_+ \neq r_-$ are reals, then the general solution of the differential equation is

$$y(x) = c_+ \, e^{r_+ x} + c_- \, e^{r_- x}.$$

The boundary conditions are

$$\left.\begin{aligned} y_0 &= y(0) = c_+ + c_- \\ y_1 &= y(L) = c_+ \, e^{c_+ L} + c_- \, e^{c_- L} \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

This system for c_+, c_- has a unique solution iff the coefficient matrix is invertible. But its determinant is

$$\begin{vmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{vmatrix} = e^{r_- L} - e^{r_+ L}.$$

Therefore, if the roots $r_+ \neq r_-$ are reals, then $e^{r_- L} \neq e^{r_+ L}$, hence there is a unique solution c_+, c_-, which in turn fixes a unique solution y of the BVP.

In the case that $r_+ = r_- = r_0$, then we have to start over, since the general solution of the differential equation is

$$y(x) = (c_1 + c_2 \, x) \, e^{r_0 x}, \qquad c_1, c_2 \in \mathbb{R}.$$

Again, the boundary conditions in Eq. (9.1.2) determine the values of the constants c_1 and c_2 as follows:

$$\left.\begin{aligned} y_0 &= y(0) = c_1 \\ y_1 &= y(L) = c_1 e^{r_0 L} + c_2 L \, e^{r_0 L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ e^{r_0 L} & L \, e^{r_0 L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

This system for c_1, c_2 has a unique solution iff the coefficient matrix is invertible. But its determinant is

$$\begin{vmatrix} 1 & 0 \\ e^{r_0 L} & L \, e^{r_0 L} \end{vmatrix} = L \, e^{r_0 L}$$

So, for $L \neq 0$ the determinant above is nonzero, then there is a unique solution c_1, c_2, which in turn fixes a unique solution y of the BVP.
Part (B): If $r_\pm = \alpha \pm i\beta$, that is complex, then

$$e^{r_\pm L} = e^{(\alpha \pm i\beta)L} = e^{\alpha L} (\cos(\beta L) \pm i \sin(\beta L)),$$

therefore

$$\begin{aligned} e^{r_- L} - e^{r_+ L} &= e^{\alpha L} \big(\cos(\beta L) - i \sin(\beta L) - \cos(\beta L) - i \sin(\beta L) \big) \\ &= -2i \, e^{\alpha L} \, \sin(\beta L). \end{aligned}$$

We conclude that

$$e^{r_- L} - e^{r_+ L} = -2i \, e^{\alpha L} \, \sin(\beta L) = 0 \quad \Leftrightarrow \quad \beta L = n\pi.$$

So for $\beta L \neq n\pi$ the BVP has a unique solution, case (Bi). But for $\beta L = n\pi$ the BVP has either no solution or infinitely many solutions, cases (Bii) and (Biii). This establishes the Theorem. □

Example 9.1.2: Find all solutions to the BVPs $y'' + y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$$

Solution: We first find the roots of the characteristic polynomial $r^2 + 1 = 0$, that is, $r_\pm = \pm i$. So the general solution of the differential equation is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$0 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 0.$$

Therefore, there is no solution.
BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_2 = 1.$$

So there is a unique solution $y(x) = \cos(x) + \sin(x)$.
BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_2 = 1.$$

Therefore, c_2 is arbitrary, so we have infinitely many solutions

$$y(x) = \cos(x) + c_2 \sin(x), \qquad c_2 \in \mathbb{R}.$$

◁

EXAMPLE 9.1.3: Find all solutions to the BVPs $y'' + 4\,y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi/4) = -1. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = -1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases}$$

SOLUTION: We first find the roots of the characteristic polynomial $r^2 + 4 = 0$, that is, $r_{\pm} = \pm 2i$. So the general solution of the differential equation is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi/4) = c_2 \quad \Rightarrow \quad c_2 = -1.$$

Therefore, there is a unique solution $y(x) = \cos(2x) - \sin(2x)$.
BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi/2) = -c_1 \quad \Rightarrow \quad c_1 = 1.$$

So, c_2 is arbitrary and we have infinitely many solutions

$$y(x) = \cos(2x) + c_2 \sin(2x), \qquad c_2 \in \mathbb{R}.$$

BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$1 = y(\pi/2) = -c_1 \quad \Rightarrow \quad c_2 = -1.$$

Therefore, we have no solution. ◁

9.1.3. **Eigenfunction Problems.** We now focus on boundary value problems that have infinitely many solutions. A particular type of these problems are called an eigenfunction problems. They are similar to the eigenvector problems we studied in § **??**. Recall that the eigenvector problem is the following: Given an $n \times n$ matrix A, find all numbers λ and nonzero vectors $\boldsymbol{v}$ solution of the algebraic linear system

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

We saw that for each λ there are infinitely many solutions $\boldsymbol{v}$, because if $\boldsymbol{v}$ is a solution so is any multiple $a\boldsymbol{v}$. An eigenfunction problem is something similar.

Definition 9.1.6. *An **eigenfunction problem** is the following: Given a linear operator* $L(y) = y'' + a_1\, y' + a_0\, y$*, find a number* λ *and a nonzero function* y *solution of*

$$L(y) = -\lambda y,$$

with homogeneous boundary conditions

$$b_1\, y(x_1) + b_2\, y'(x_1) = 0,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = 0.$$

Remarks:

- Notice that $y = 0$ is always a solution of the BVP above.
- Eigenfunctions are the nonzero solutions of the BVP above.
- Hence, the eigenfunction problem is a BVP with infinitely many solutions.
- So, we look for λ such that the operator $L(y) + \lambda\, y$ has characteristic polynomial with complex roots.
- So, λ is such that $L(y) + \lambda\, y$ has oscillatory solutions.
- Our examples focus on the linear operator $L(y) = y''$.

Example 9.1.4: Find all numbers λ and nonzero functions y solutions of the BVP

$$y'' + \lambda\, y = 0, \quad \text{with} \quad y(0) = 0, \quad y(L) = 0, \qquad L > 0.$$

Solution: We divide the problem in three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.
Case (a): $\lambda = -\mu^2 < 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is $y = c_+\, e^{\mu x} + c_-\, e^{-\mu x}$. The BC imply

$$0 = y(0) = c_+ + c_-, \qquad 0 = y(L) = c_+\, e^{\mu L} + c_-\, e^{-\mu L}.$$

So from the first equation we get $c_+ = -c_-$, so

$$0 = -c_-\, e^{\mu L} + c_-\, e^{-\mu L} \quad \Rightarrow \quad -c_-(e^{\mu L} - e^{-\mu L}) = 0 \quad \Rightarrow \quad c_- = 0, \quad c_+ = 0.$$

So the only the solution is $y = 0$, then there are no eigenfunctions with negative eigenvalues.
Case (b): $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y = c_0 + c_1 x.$$

The BC imply

$$0 = y(0) = c_0, \qquad 0 = y(L) = c_1 L \quad \Rightarrow \quad c_1 = 0.$$

So the only solution is $y = 0$, then there are no eigenfunctions with eigenvalue $\lambda = 0$.
Case (c): $\lambda = \mu^2 > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y = c_+ \cos(\mu x) + c_- \sin(\mu x)$. The BC imply

$$0 = y(0) = c_+, \qquad 0 = y(L) = c_+ \cos(\mu L) + c_- \sin(\mu L).$$

Since $c_+ = 0$, the second equation above is

$$c_- \sin(\mu L) = 0, \qquad c_- \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi.$$

So we get $\mu_n = n\pi/L$, hence the eigenvalue eigenfunction pairs are

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

Since we need only one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \qquad n \geqslant 1.$$

$\lhd$

Example 9.1.5: Find the numbers λ and the nonzero functions y solutions of the BVP

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(L) = 0, \qquad L > 0.$$

Solution: We divide the problem in three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.
Case (a): Let $\lambda = -\mu^2$, with $\mu > 0$, so the equation is $y'' - \mu^2\, y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu,$$

The general solution is $y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 0 &= y'(L) = -\mu c_1 e^{-\mu L} + \mu c_2 e^{\mu L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{vmatrix} = \mu\big(e^{\mu L} + e^{-\mu L}\big) \neq 0.$$

So, the linear system above for c_1, c_2 has a unique solution $c_1 = c_2 = 0$. Hence, we get the only solution $y = 0$. This means there are no eigenfunctions with negative eigenvalues.
Case (b): Let $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$0 = y(0) = c_1, \qquad 0 = y'(L) = c_2.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 0$.
Case (c): Let $\lambda = \mu^2$, with $\mu > 0$, so the equation is $y'' + \mu^2\, y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu\, i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. The BC imply

$$\left.\begin{aligned} &0 = y(0) = c_1, \\ &0 = y'(L) = -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \cos(\mu L) = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\cos(\mu L) = 0 \quad \Leftrightarrow \quad \mu_n L = (2n-1)\frac{\pi}{2}, \qquad n \geqslant 1.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = c_n \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

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Example 9.1.6: Find the numbers λ and the nonzero functions y solutions of the BVP

$$x^2\,y'' - x\,y' = -\lambda\,y, \qquad y(1) = 0, \quad y(\ell) = 0, \quad \ell > 1.$$

Solution: Let us rewrite the equation as

$$x^2\,y'' - x\,y' + \lambda y = 0.$$

This is an Euler equidimensional equation. From § **??** we know we need to look for the solutions $r_\pm$ of the indicial polynomial

$$r(r-1) - r + \lambda = 0 \quad \Rightarrow \quad r^2 - 2r + \lambda = 0 \quad \Rightarrow \quad r_\pm = 1 \pm \sqrt{1-\lambda}.$$

Case (a): Let $1-\lambda = 0$, so we have a repeated root $r_+ = r_- = 1$. The general solution to the differential equation is

$$y(x) = \big(c_1 + c_2 \ln(x)\big)\, x.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = \big(c_1 + c_2\ln(\ell)\big)\,\ell \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \ln(\ell) = 0 \quad \Rightarrow \quad c_2 = 0.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 1$.

Case (b): Let $1-\lambda > 0$, so we can rewrite it as $1-\lambda = \mu^2$, with $\mu > 0$. Then, $r_\pm = 1 \pm \mu$, and so the general solution to the differential equation is given by

$$y(x) = c_1 x^{(1-\mu)} + c_2 x^{(1+\mu)},$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2, \\ 0 &= y(\ell) = c_1 \ell^{(1-\mu)} + c_2 \ell^{(1+\mu)} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{vmatrix} = \ell\,\big(\ell^{\mu} - \ell^{-\mu}\big) \neq 0 \quad \Leftrightarrow \quad \ell \neq \pm 1.$$

Since $\ell > 1$, the matrix above is invertible, and the linear system for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$. Hence we get the only solution $y = 0$. This means there are no eigenfunctions with eigenvalues $\lambda < 1$.

Case (c): Let $1-\lambda < 0$, so we can rewrite it as $1-\lambda = -\mu^2$, with $\mu > 0$. Then $r_\pm = 1 \pm i\mu$, and so the general solution to the differential equation is

$$y(x) = x\big[c_1 \cos\big(\mu \ln(x)\big) + c_2 \sin\big(\mu \ln(x)\big)\big].$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = c_1 \ell \cos\big(\mu \ln(\ell)\big) + c_2 \ell \sin\big((\mu \ln(\ell)\big) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \sin\big(\mu \ln(\ell)\big) = 0.$$

Since we are interested in nonzero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\sin\big(\mu \ln(\ell)\big) = 0 \quad \Leftrightarrow \quad \mu_n \ln(\ell) = n\pi, \qquad n \geqslant 1.$$

Recalling that $1 - \lambda_n = -\mu_n^2$, we get $\lambda_n = 1 + \mu_n^2$, hence,

$$\lambda_n = 1 + \frac{n^2\pi^2}{\ln^2(\ell)}, \qquad y_n(x) = c_n x \sin\Big(\frac{n\pi \ln(x)}{\ln(\ell)}\Big), \qquad n \geqslant 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = 1 + \frac{n^2\pi^2}{\ln^2(\ell)}, \qquad y_n(x) = x \sin\Big(\frac{n\pi \ln(x)}{\ln(\ell)}\Big), \qquad n \geqslant 1.$$

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