## 6.5. Diagonalizable Matrices

It is useful to introduce few more concepts, that are common in the literature.

**Definition 6.5.1.** *The* ***characteristic polynomial*** *of an $n \times n$ matrix $A$ is the function*

$$p(\lambda) = \det(A - \lambda I).$$

Example 6.5.1: Find the characteristic polynomial of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We need to compute the determinant

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9.$$

We conclude that the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda - 8$. ◁

Since the matrix $A$ in this example is $2 \times 2$, its characteristic polynomial has degree two. One can show that the characteristic polynomial of an $n \times n$ matrix has degree $n$. The eigenvalues of the matrix are the roots of the characteristic polynomial. Different matrices may have different types of roots, so we try to classify these roots in the following definition.

**Definition 6.5.2.** *Given an $n \times n$ matrix $A$ with real eigenvalues $\lambda_i$, where $i = 1, \cdots, k \leqslant n$, it is always possible to express the characteristic polynomial of $A$ as*

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

*The number $r_i$ is called the* ***algebraic multiplicity*** *of the eigenvalue $\lambda_i$. Furthermore, the* ***geometric multiplicity*** *of an eigenvalue $\lambda_i$, denoted as $s_i$, is the maximum number of eigenvectors of $\lambda_i$ that form a linearly independent set.*

Example 6.5.2: Find the eigenvalues algebraic and geometric multiplicities of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Solution: In order to find the algebraic multiplicity of the eigenvalues we need first to find the eigenvalues. We now that the characteristic polynomial of this matrix is given by

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9.$$

The roots of this polynomial are $\lambda_1 = 4$ and $\lambda_2 = -2$, so we know that $p(\lambda)$ can be rewritten in the following way,

$$p(\lambda) = (\lambda - 4)(\lambda + 2).$$

We conclude that the algebraic multiplicity of the eigenvalues are both one, that is,

$$\lambda_1 = 4, \quad r_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad r_2 = 1.$$

In order to find the geometric multiplicities of matrix eigenvalues we need first to find the matrix eigenvectors. This part of the work was already done in the Example **??** above and the result is

$$\lambda_1 = 4, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

From this expression we conclude that the geometric multiplicities for each eigenvalue are just one, that is,

$$\lambda_1 = 4, \quad s_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad s_2 = 1.$$

◁

The following example shows that two matrices can have the same eigenvalues, and so the same algebraic multiplicities, but different eigenvectors with different geometric multiplicities.

Example 6.5.3: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: We start finding the eigenvalues, the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = -\dfrac{v_3^{(1)}}{2}, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 2$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$, which are all solutions of the linear system

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} \quad \text{free,} \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain two linearly independent solutions, the first one $\boldsymbol{v}^{(2)}$ with the choice $v_1^{(2)} = 1$, $v_2^{(2)} = 0$, and the second one $\boldsymbol{w}^{(2)}$ with the choice $v_1^{(2)} = 0$, $v_2^{(2)} = 1$, that is

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 2.$$

Summarizing, the matrix in this example has three linearly independent eigenvectors. $\triangleleft$

EXAMPLE 6.5.4: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

SOLUTION: Notice that this matrix has only the coefficient $a_{12}$ different from the previous example. Again, we start finding the eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

So this matrix has the same eigenvalues and algebraic multiplicities as the matrix in the previous example. We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = 0, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 1$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$. However, in this case we obtain only one solution, as this calculation shows,

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} = 0, \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain only one linearly independent solution, which corresponds to the choice $v_1^{(2)} = 1$, that is,

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 1.$$

Summarizing, the matrix in this example has only two linearly independent eigenvectors, and in the case of the eigenvalue $\lambda_2 = 3$ we have the strict inequality

$$1 = s_2 < r_2 = 2.$$

$\triangleleft$

We first introduce the notion of a diagonal matrix. Later on we define a diagonalizable matrix as a matrix that can be transformed into a diagonal matrix by a simple transformation.

**Definition 6.5.3.** *An $n \times n$ matrix $A$ is called* ***diagonal*** *iff* $A = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}$.

That is, a matrix is diagonal iff every nondiagonal coefficient vanishes. From now on we use the following notation for a diagonal matrix $A$:

$$A = \operatorname{diag}\big[a_{11}, \cdots, a_{nn}\big] = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}.$$

This notation says that the matrix is diagonal and shows only the diagonal coefficients, since any other coefficient vanishes. The next result says that the eigenvalues of a diagonal matrix are the matrix diagonal elements, and it gives the corresponding eigenvectors.

**Theorem 6.5.4.** *If $D = \operatorname{diag}[d_{11}, \cdots, d_{nn}]$, then eigenpairs of $D$ are*

$$\lambda_1 = d_{11}, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \lambda_n = d_{nn}, \quad \boldsymbol{v}^{(n)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Diagonal matrices are simple to manipulate since they share many properties with numbers. For example the product of two diagonal matrices is commutative. It is simple to compute power functions of a diagonal matrix. It is also simple to compute more involved functions of a diagonal matrix, like the exponential function.

EXAMPLE 6.5.5: For every positive integer $n$ find $A^n$, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

SOLUTION: We start computing $A^2$ as follows,

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}.$$

We now compute $A^3$,

$$A^3 = A^2 A = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix}.$$

Using induction, it is simple to see that $A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$. ◁

Many properties of diagonal matrices are shared by diagonalizable matrices. These are matrices that can be transformed into a diagonal matrix by a simple transformation.

**Definition 6.5.5.** *An $n \times n$ matrix $A$ is called* ***diagonalizable*** *iff there exists an invertible matrix $P$ and a diagonal matrix $D$ such that*

$$A = PDP^{-1}.$$

**Remarks:**

(a) Systems of linear differential equations are simple to solve in the case that the coefficient matrix is diagonalizable. One decouples the differential equations, solves the decoupled equations, and transforms the solutions back to the original unknowns.
(b) Not every square matrix is diagonalizable. For example, matrix $A$ below is diagonalizable while $B$ is not,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}.$$

Example 6.5.6: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable, where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Solution: That matrix $P$ is invertible can be verified by computing its determinant, $\det(P) = 1 - (-1) = 2$. Since the determinant is nonzero, $P$ is invertible. Using linear algebra methods one can find out that the inverse matrix is $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Now we only need to verify that $PDP^{-1}$ is indeed $A$. A straightforward calculation shows

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A. \end{aligned}$$

$\lhd$

There is a deep relation between the eigenpairs of a matrix and whether that matrix is diagonalizable.

**Theorem 6.5.6** (Diagonalizable Matrix). *An $n \times n$ matrix $A$ is diagonalizable iff $A$ has a linearly independent set of $n$ eigenvectors. Furthermore, if $\lambda_i, \boldsymbol{v}_i$, for $i = 1, \cdots, n$, are eigenpairs of $A$, then*

$$A = PDP^{-1}, \quad P = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n], \quad D = \operatorname{diag}\left[\lambda_1, \cdots, \lambda_n\right].$$

**Proof of Theorem 6.5.6:**

($\Rightarrow$) Since matrix $A$ is diagonalizable, there exist an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Multiply this equation by $P^{-1}$ on the left and by $P$ on the right, we get

$$D = P^{-1}AP. \tag{6.5.1}$$

Since $n \times n$ matrix $D$ is diagonal, it has a linearly independent set of $n$ eigenvectors, given by the column vectors of the identity matrix, that is,

$$D\boldsymbol{e}^{(i)} = d_{ii}\boldsymbol{e}^{(i)}, \qquad D = \operatorname{diag}\left[d_{11}, \cdots, d_{nn}\right], \qquad I = \left[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\right].$$

So, the pair $d_{ii}, \boldsymbol{e}^{(i)}$ is an eigenvalue-eigenvector pair of $D$, for $i = 1 \cdots, n$. Using this information in Eq. (6.5.1) we get

$$d_{ii}\, \boldsymbol{e}^{(i)} = D\boldsymbol{e}^{(i)} = P^{-1}AP\boldsymbol{e}^{(i)} \quad \Rightarrow \quad A\big(P\boldsymbol{e}^{(i)}\big) = d_{ii}\, \big(P\boldsymbol{e}^{(i)}\big),$$

where the last equation comes from multiplying the former equation by $P$ on the left. This last equation says that the vectors $\boldsymbol{v}^{(i)} = P\boldsymbol{e}^{(i)}$ are eigenvectors of $A$ with eigenvalue $d_{ii}$. By definition, $\boldsymbol{v}^{(i)}$ is the $i$-th column of matrix $P$, that is,

$$P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big].$$

Since matrix $P$ is invertible, the eigenvectors set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. This establishes this part of the Theorem.

($\Leftarrow$) Let $\lambda_i, \boldsymbol{v}^{(i)}$ be eigenvalue-eigenvector pairs of matrix $A$, for $i = 1, \cdots, n$. Now use the eigenvectors to construct matrix $P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big]$. This matrix is invertible, since the eigenvector set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We now show that matrix $P^{-1}AP$ is diagonal. We start computing the product

$$AP = A\big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big] = \big[A\boldsymbol{v}^{(1)}, \cdots, A\boldsymbol{v}^{(n)}\big], = \big[\lambda_1\boldsymbol{v}^{(1)} \cdots, \lambda_n\boldsymbol{v}^{(n)}\big].$$

that is,

$$P^{-1}AP = P^{-1}\big[\lambda_1\boldsymbol{v}^{(1)}, \cdots, \lambda_n\boldsymbol{v}^{(n)}\big] = \big[\lambda_1 P^{-1}\boldsymbol{v}^{(1)}, \cdots, \lambda_n P^{-1}\boldsymbol{v}^{(n)}\big].$$

At this point it is useful to recall that $P^{-1}$ is the inverse of $P$,

$$I = P^{-1}P \quad \Leftrightarrow \quad \big[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\big] = P^{-1}\big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big] = \big[P^{-1}\boldsymbol{v}^{(1)}, \cdots, P^{-1}\boldsymbol{v}^{(n)}\big].$$

So, $\boldsymbol{e}^{(i)} = P^{-1}\boldsymbol{v}^{(i)}$, for $i = 1 \cdots, n$. Using these equations in the equation for $P^{-1}AP$,

$$P^{-1}AP = \big[\lambda_1\boldsymbol{e}^{(1)}, \cdots, \lambda_n\boldsymbol{e}^{(n)}\big] = \mathrm{diag}\,\big[\lambda_1, \cdots, \lambda_n\big].$$

Denoting $D = \mathrm{diag}\,\big[\lambda_1, \cdots, \lambda_n\big]$ we conclude that $P^{-1}AP = D$, or equivalently

$$A = PDP^{-1}, \qquad P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big], \qquad D = \mathrm{diag}\,\big[\lambda_1, \cdots, \lambda_n\big].$$

This means that $A$ is diagonalizable. This establishes the Theorem. □

EXAMPLE 6.5.7: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

SOLUTION: We know that the eigenvalue-eigenvector pairs are

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce $P$ and $D$ as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

We must show that $A = PDP^{-1}$. This is indeed the case, since

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We conclude, $PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A$, that is, $A$ is diagonalizable. ◁

With Theorem 6.5.6 we can show that a matrix *is not* diagonalizable.

Example 6.5.8: Show that matrix $B = \dfrac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We first compute the matrix eigenvalues. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(\frac{3}{2} - \lambda\right) & \frac{1}{2} \\ -\frac{1}{2} & \left(\frac{5}{2} - \lambda\right) \end{vmatrix} = \left(\frac{3}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) + \frac{1}{4} = \lambda^2 - 4\lambda + 4.$$

The roots of the characteristic polynomial are computed in the usual way,

$$\lambda = \frac{1}{2}\big[4 \pm \sqrt{16 - 16}\big] \quad \Rightarrow \quad \lambda = 2, \qquad r = 2.$$

We have obtained a single eigenvalue with algebraic multiplicity $r = 2$. The associated eigenvectors are computed as the solutions to the equation $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$. Then,

$$(A - 2I) = \begin{bmatrix} \left(\frac{3}{2} - 2\right) & \frac{1}{2} \\ -\frac{1}{2} & \left(\frac{5}{2} - 2\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s = 1.$$

We conclude that the biggest linearly independent set of eigenvalues for the $2 \times 2$ matrix $B$ contains only one vector, insted of two. Therefore, matrix $B$ is not diagonalizable. $\triangleleft$

Theorem 6.5.6 shows the importance of knowing whether an $n \times n$ matrix has a linearly independent set of $n$ eigenvectors. However, more often than not, there is no simple way to check this property other than to compute all the matrix eigenvectors. But there is a simpler particular case, the case when an $n \times n$ matrix has $n$ *different* eigenvalues. Then, we do not need to compute the eigenvectors. The following result says that such matrix always have a linearly independent set of $n$ eigenvectors, hence, by Theorem 6.5.6, it is diagonalizable.

**Theorem 6.5.7** (Different Eigenvalues)**.** *If an $n \times n$ matrix has $n$ different eigenvalues, then this matrix has a linearly independent set of $n$ eigenvectors.*

**Proof of Theorem 6.5.7:** Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix $A$, all different from each other. Let $\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}$ the corresponding eigenvectors, that is, $A\boldsymbol{v}^{(i)} = \lambda_i \boldsymbol{v}^{(i)}$, with $i = 1, \cdots, n$. We have to show that the set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We assume that the opposite is true and we obtain a contradiction. Let us assume that the set above is linearly dependent, that is, there are constants $c_1, \cdots, c_n$, not all zero, such that,

$$c_1 \boldsymbol{v}^{(1)} + \cdots + c_n \boldsymbol{v}^{(n)} = \boldsymbol{0}. \tag{6.5.2}$$

Let us name the eigenvalues and eigenvectors such that $c_1 \neq 0$. Now, multiply the equation above by the matrix $A$, the result is,

$$c_1 \lambda_1 \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Multiply Eq. (6.5.2) by the eigenvalue $\lambda_n$, the result is,

$$c_1 \lambda_n \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Subtract the second from the first of the equations above, then the last term on the right-hand sides cancels out, and we obtain,

$$c_1(\lambda_1 - \lambda_n)\boldsymbol{v}^{(1)} + \cdots + c_{n-1}(\lambda_{n-1} - \lambda_n)\boldsymbol{v}^{(n-1)} = \boldsymbol{0}. \tag{6.5.3}$$

Repeat the whole procedure starting with Eq. (6.5.3), that is, multiply this later equation by matrix $A$ and also by $\lambda_{n-1}$, then subtract the second from the first, the result is,

$$c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\boldsymbol{v}^{(1)} + \cdots + c_{n-2}(\lambda_{n-2} - \lambda_n)(\lambda_{n-2} - \lambda_{n-1})\boldsymbol{v}^{(n-2)} = \boldsymbol{0}.$$

Repeat the whole procedure a total of $n-1$ times, in the last step we obtain the equation

$$c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\cdots(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)\boldsymbol{v}^{(1)} = \boldsymbol{0}.$$

Since all the eigenvalues are different, we conclude that $c_1 = 0$, however this contradicts our assumption that $c_1 \neq 0$. Therefore, the set of $n$ eigenvectors must be linearly independent. This establishes the Theorem. □

Example 6.5.9: Is matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ diagonalizable?

Solution: We compute the matrix eigenvalues, starting with the characteristic polynomial,

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda \quad \Rightarrow \quad p(\lambda) = \lambda(\lambda - 2).$$

The roots of the characteristic polynomial are the matrix eigenvalues,

$$\lambda_1 = 0, \qquad \lambda_2 = 2.$$

The eigenvalues are different, so by Theorem 6.5.7, matrix $A$ is diagonalizable. ◁

6.5.1. **Exercises.**

**6.5.1.-** .

**6.5.2.-** .