We introduce a generalized function—the Dirac delta. We define the Dirac delta as a limit $n \to \infty$ of a particular sequence of functions, $\{\delta_n\}$. We will see that this limit is a function on the domain $\mathbb{R} - \{0\}$, but it is not a function on $\mathbb{R}$. For that reason we call this limit a generalized function—the Dirac delta generalized function.

We will show that each element in the sequence $\{\delta_n\}$ has a Laplace transform, and this sequence of Laplace transforms $\{\mathcal{L}[\delta_n]\}$ has a limit as $n \to \infty$. We use this limit of Laplace transforms to define the Laplace transform of the Dirac delta.

We will solve differential equations having the Dirac delta generalized function as source. Such differential equations appear often when one describes physical systems with impulsive forces—forces acting on a very short time but transfering a finite momentum to the system. Dirac's delta is tailored to model impulsive forces.

4.4.1. **Sequence of Functions and the Dirac Delta.** A sequence of functions is a sequence whose elements are functions. If each element in the sequence is a continuous function, we say that this is a sequence of continuous functions. Given a sequence of functions $\{y_n\}$, we compute the $\lim_{n\to\infty} y_n(t)$ for a fixed t. The limit depends on t, so it is a function of t, and we write it as

$$\lim_{n\to\infty} y_n(t) = y(t).$$

The domain of the limit function y is smaller or equal to the domain of the y_n. The limit of a sequence of continuous functions may or may not be a continuous function.

Example 4.4.1: The limit of the sequence below is a continuous function,

$$\Big\{ f_n(t) = \sin\Big(\Big(1 + \frac{1}{n}\Big)t\Big)\Big\} \to \sin(t) \quad \text{as} \quad n \to \infty.$$

As usual in this section, the limit is computed for each fixed value of t. ◁

However, not every sequence of continuous functions has a continuous function as a limit.

Example 4.4.2: Consider now the following sequence, $\{u_n\}$, for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases} \tag{4.4.1}$$

This is a sequence of continuous functions whose limit is a discontinuous function. From the few graphs in Fig. 1 we can see that the limit $n \to \infty$ of the sequence above is a step function, indeed, $\lim_{n\to\infty} u_n(t) = \tilde{u}(t)$, where

$$\tilde{u}(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

We used a tilde in the name $\tilde{u}$ because this step function is not the same we defined in the previous section. The step u in § **??** satisfied $u(0) = 1$. ◁

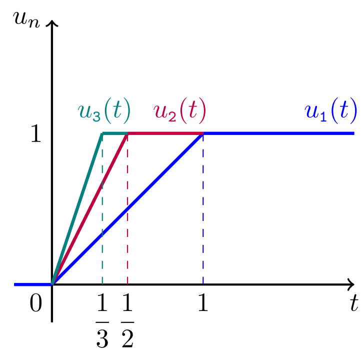

Figure 1. A few functions in the sequence $\{u_n\}$.

Exercise: Find a sequence $\{u_n\}$ so that its limit is the step function u defined in § **??**.

Although every function in the sequence $\{u_n\}$ is continuous, the limit $\tilde{u}$ is a discontinuous function. It is not difficult to see that one can construct sequences of continuous functions having no limit at all. A similar situation happens when one considers sequences of piecewise discontinuous functions. In this case the limit could be a continuous function, a piecewise discontinuous function, or not a function at all.

We now introduce a particular sequence of piecewise discontinuous functions with domain $\mathbb{R}$ such that the limit as $n \to \infty$ does not exist for all values of the independent variable t. The limit of the sequence is not a function with domain $\mathbb{R}$. In this case, the limit is a new type of object that we will call Dirac's delta generalized function. Dirac's delta is the limit of a sequence of particular bump functions.

Definition 4.4.1. *The **Dirac delta** generalized function is the limit*

$$\delta(t) = \lim_{n\to\infty} \delta_n(t),$$

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

$$\delta_n(t) = n\Big[u(t) - u\big(t - \frac{1}{n}\big)\Big]. \tag{4.4.2}$$

The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t < \dfrac{1}{n} \\ 0, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0, \\ \infty & \text{for} \quad t = 0. \end{cases}$$

Remark: It can be shown that there exist infinitely many sequences $\{\tilde{\delta}_n\}$ such that their limit as $n \to \infty$ is Dirac's delta. For example, another sequence is

$$\tilde{\delta}_n(t) = n\Big[u\big(t + \frac{1}{2n}\big) - u\big(t - \frac{1}{2n}\big)\Big]$$
$$= \begin{cases} 0, & t < -\dfrac{1}{2n} \\ n, & -\dfrac{1}{2n} \leqslant t \leqslant \dfrac{1}{2n} \\ 0, & t > \dfrac{1}{2n}. \end{cases}$$

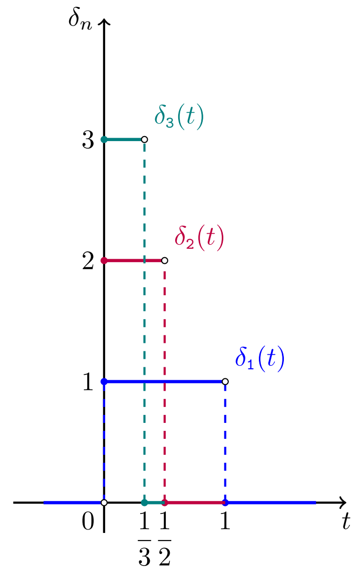

FIGURE 2. A few functions in the sequence $\{\delta_n\}$.

The Dirac delta generalized function is the function identically zero on the domain $\mathbb{R}-\{0\}$. Dirac's delta is not defined at $t = 0$, since the limit diverges at that point. If we shift each element in the sequence by a real number c, then we define

$$\delta(t - c) = \lim_{n\to\infty} \delta_n(t - c), \qquad c \in \mathbb{R}.$$

This shifted Dirac's delta is identically zero on $\mathbb{R} - \{c\}$ and diverges at $t = c$. If we shift the graphs given in Fig. 2 by any real number c, one can see that

$$\int_c^{c+1} \delta_n(t - c)\, dt = 1$$

for every $n \geqslant 1$. Therefore, the sequence of integrals is the constant sequence, $\{1, 1, \cdots\}$, which has a trivial limit, 1, as $n \to \infty$. This says that the divergence at $t = c$ of the sequence $\{\delta_n\}$ is of a very particular type. The area below the graph of the sequence elements is always the same. We can say that this property of the sequence provides the main defining property of the Dirac delta generalized function.

Using a limit procedure one can generalize several operations from a sequence to its limit. For example, translations, linear combinations, and multiplications of a function by a generalized function, integration and Laplace transforms.

Definition 4.4.2. *We introduce the following operations on the Dirac delta:*

$$f(t)\,\delta(t-c) + g(t)\,\delta(t-c) = \lim_{n\to\infty} \big[f(t)\,\delta_n(t-c) + g(t)\,\delta_n(t-c)\big],$$

$$\int_a^b \delta(t-c)\,dt = \lim_{n\to\infty} \int_a^b \delta_n(t-c)\,dt,$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)].$$

Remark: The notation in the definitions above could be misleading. In the left hand sides above we use the same notation as we use on functions, although Dirac's delta is not a function on $\mathbb{R}$. Take the integral, for example. When we integrate a function f, the integration symbol means "take a limit of Riemann sums", that is,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\,\Delta x, \qquad x_i = a + i\,\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

However, when f is a generalized function in the sense of a limit of a sequence of functions $\{f_n\}$, then by the integration symbol we mean to compute a different limit,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b f_n(t)\,dt.$$

We use the same symbol, the integration, to mean two different things, depending whether we integrate a function or a generalized function. This remark also holds for all the operations we introduce on generalized functions, specially the Laplace transform, that will be often used in the rest of this section.

4.4.2. **Computations with the Dirac Delta.** Once we have the definitions of operations involving the Dirac delta, we can actually compute these limits. The following statement summarizes few interesting results. The first formula below says that the infinity we found in the definition of Dirac's delta is of a very particular type; that infinity is such that Dirac's delta is integrable, in the sense defined above, with integral equal one.

Theorem 4.4.3. *For every $c \in \mathbb{R}$ and $\epsilon > 0$ holds,* $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = 1.$

Proof of Theorem 4.4.3: The integral of a Dirac's delta generalized function is computed as a limit of integrals,

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} \int_{c-\epsilon}^{c+\epsilon} \delta_n(t-c)\,dt.$$

If we choose $n > 1/\epsilon$, equivalently $1/n < \epsilon$, then the domain of the functions in the sequence is inside the interval $(c-\epsilon, c+\epsilon)$, and we can write

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} \int_{c}^{c+\frac{1}{n}} n\,dt, \qquad \text{for} \qquad \frac{1}{n} < \epsilon.$$

Then it is simple to compute

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} n\Big(c + \frac{1}{n} - c\Big) = \lim_{n\to\infty} 1 = 1.$$

This establishes the Theorem. □

The next result is also deeply related with the defining property of the Dirac delta—the sequence functions have all graphs of unit area.

Theorem 4.4.4. *If f is continuous on (a,b) and $c \in (a,b)$, then* $\displaystyle\int_a^b f(t)\,\delta(t-c)\,dt = f(c)$.

Proof of Theorem 4.4.4: We again compute the integral of a Dirac's delta as a limit of a sequence of integrals,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} \int_a^b \delta_n(t-c)\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_a^b n\Big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\Big]\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\,f(t)\,dt, \qquad \frac{1}{n} < (b-c),
\end{aligned}$$

To get the last line we used that $c \in [a,b]$. Let F be any primitive of f, so $F(t) = \int f(t)\,dt$. Then we can write,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} n\big[F\big(c+\frac{1}{n}\big) - F(c)\big] \\
&= \lim_{n\to\infty} \frac{1}{\big(\frac{1}{n}\big)}\big[F\big(c+\frac{1}{n}\big) - F(c)\big] \\
&= F'(c) \\
&= f(c).
\end{aligned}$$

This establishes the Theorem. □

In our next result we compute the Laplace transform of the Dirac delta. We give two proofs of this result. In the first proof we use the previous theorem. In the second proof we use the same idea used to prove the previous theorem.

Theorem 4.4.5. *For all $s \in \mathbb{R}$ holds* $\mathcal{L}[\delta(t-c)] = \begin{cases} e^{-cs} & \text{for } c \geqslant 0, \\ 0 & \text{for } c < 0. \end{cases}$

First Proof of Theorem 4.4.5: We use the previous theorem on the integral that defines a Laplace transform. Although the previous theorem applies to definite integrals, not to improper integrals, it can be extended to cover improper integrals. In this case we get

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\,\delta(t-c)\,dt = \begin{cases} e^{-cs} & \text{for } c \geqslant 0, \\ 0 & \text{for } c < 0, \end{cases}$$

This establishes the Theorem. □

Second Proof of Theorem 4.4.5: The Laplace transform of a Dirac's delta is computed as a limit of Laplace transforms,

$$\begin{aligned}\mathcal{L}[\delta(t-c)] &= \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \\ &= \lim_{n\to\infty} \mathcal{L}\Big[n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\Big] \\ &= \lim_{n\to\infty} \int_0^\infty n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\, e^{-st}\, dt.\end{aligned}$$

The case $c < 0$ is simple. For $\dfrac{1}{n} < |c|$ holds

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_0^\infty 0\, dt \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 0, \qquad \text{for } s \in \mathbb{R}, \quad c < 0.$$

Consider now the case $c \geqslant 0$. We then have,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, e^{-st}\, dt.$$

For $s = 0$ we get

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, dt = 1 \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 1 \quad \text{for } s = 0, \quad c \geqslant 0.$$

In the case that $s \neq 0$ we get,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, e^{-st}\, dt = \lim_{n\to\infty} -\frac{n}{s}\big(e^{-cs} - e^{-(c+\frac{1}{n})s}\big) = e^{-cs} \lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\dfrac{s}{n}\right)}.$$

The limit on the last line above is a singular limit of the form $\frac{0}{0}$, so we can use the l'Hôpital rule to compute it, that is,

$$\lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\dfrac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\dfrac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\dfrac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We then obtain,

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \quad \text{for } s \neq 0, \quad c \geqslant 0.$$

This establishes the Theorem. □

4.4.3. **Applications of the Dirac Delta.** Dirac's delta generalized functions describe *impulsive forces* in mechanical systems, such as the force done by a stick hitting a marble. An impulsive force acts on an infinitely short time and transmits a finite momentum to the system.

Example 4.4.3: Use Newton's equation of motion and Dirac's delta to describe the change of momentum when a particle is hit by a hammer.

Solution: A point particle with mass m, moving on one space direction, x, with a force F acting on it is described by

$$ma = F \quad \Leftrightarrow \quad mx''(t) = F(t, x(t)),$$

where $x(t)$ is the particle position as function of time, $a(t) = x''(t)$ is the particle acceleration, and we will denote $v(t) = x'(t)$ the particle velocity. We saw in § **??** that Newton's second law of motion is a second order differential equation for the position function x. Now it is more convenient to use the *particle momentum*, $p = mv$, to write the Newton's equation,

$$mx'' = mv' = (mv)' = F \quad \Rightarrow \quad p' = F.$$

So the force F changes the momentum, P. If we integrate on an interval $[t_1, t_2]$ we get

$$\Delta p = p(t_2) - p(t_1) = \int_{t_1}^{t_2} F(t, x(t))\, dt.$$

Suppose that an impulsive force is acting on a particle at t_0 transmitting a finite momentum, say p_0. This is where the Dirac delta is uselful for, because we can write the force as

$$F(t) = p_0\, \delta(t - t_0),$$

then $F = 0$ on $\mathbb{R} - \{t_0\}$ and the momentum transferred to the particle by the force is

$$\Delta p = \int_{t_0-\Delta t}^{t_0+\Delta t} p_0\, \delta(t - t_0)\, dt = p_0.$$

The momentum tranferred is $\Delta p = p_0$, but the force is identically zero on $\mathbb{R} - \{t_0\}$. We have transferred a finite momentum to the particle by an interaction at a single time t_0. $\lhd$

4.4.4. **The Impulse Response Function.** We now want to solve differential equations with the Dirac delta as a source. But there is a particular type of solutions that will be important later on—solutions to initial value problems with the Dirac delta source and zero initial conditions. We give these solutions a particular name.

Definition 4.4.6. *The* ***impulse response function*** *at the point $c \geqslant 0$ of the constant coefficients linear operator $L(y) = y'' + a_1\, y' + a_0\, y$, is the solution y_δ of*

$$L(y_\delta) = \delta(t - c), \qquad y_\delta(0) = 0, \qquad y_\delta'(0) = 0.$$

Remark: Impulse response functions are also called *fundamental solutions*.

Theorem 4.4.7. *The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds*

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

where p is the characteristic polynomial of L.

Remark: The impulse response function y_δ at $c = 0$ satifies

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Proof of Theorem 4.4.7: Compute the Laplace transform of the differential equation for for the impulse response function y_δ,

$$\mathcal{L}[y''] + a_1\, \mathcal{L}[y'] + a_0\, \mathcal{L}[y] = \mathcal{L}[\delta(t - c)] = e^{-cs}.$$

Since the initial data for y_δ is trivial, we get

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] = e^{-cs}.$$

Since $p(s) = s^2 + a_1 s + a_0$ is the characteristic polynomial of L, we get

$$\mathcal{L}[y] = \frac{e^{-cs}}{p(s)} \quad \Leftrightarrow \quad y(t) = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

All the steps in this calculation are if and only ifs. This establishes the Theorem. $\square$

EXAMPLE 4.4.4: Find the impulse response function at $t = 0$ of the linear operator

$$L(y) = y'' + 2y' + 2y.$$

SOLUTION: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2y_\delta' + 2y_\delta = \delta(t), \qquad y_\delta(0) = 0, \qquad y_\delta'(0) = 0.$$

Since the souce is a Dirac delta, we have to use the Laplace transform to solve this problem. So we compute the Laplace transform on both sides of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t)] = 1 \quad \Rightarrow \quad (s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = 1,$$

where we have introduced the initial conditions on the last equation above. So we obtain

$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)}.$$

The denominator in the equation above has complex valued roots, since

$$s_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4 - 8}\big],$$

therefore, we complete squares $s^2 + 2s + 2 = (s + 1)^2 + 1$. We need to solve the equation

$$\mathcal{L}[y_\delta] = \frac{1}{\big[(s+1)^2 + 1\big]} = \mathcal{L}[e^{-t}\sin(t)] \quad \Rightarrow \quad y_\delta(t) = e^{-t}\sin(t).$$

◁

EXAMPLE 4.4.5: Find the impulse response function at $t = c \geqslant 0$ of the linear operator

$$L(y) = y'' + 2\,y' + 2\,y.$$

SOLUTION: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2\,y_\delta' + 2\,y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

We have to use the Laplace transform to solve this problem because the source is a Dirac's delta generalized function. So, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t - c)].$$

Since the initial conditions are all zero and $c \geqslant 0$, we get

$$(s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4 - 8}\right]$$

The denominator has complex roots. Then, it is convenient to complete the square in the denominator,

$$s^2 + 2s + 2 = \left[s^2 + 2\Big(\frac{2}{2}\Big)\,s + 1\right] - 1 + 2 = (s + 1)^2 + 1.$$

Therefore, we obtain the expression,

$$\mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s + 1)^2 + 1}.$$

Recall that $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2 + 1}$, and $\mathcal{L}[f](s - c) = \mathcal{L}[e^{ct}\,f(t)]$. Then,

$$\frac{1}{(s + 1)^2 + 1} = \mathcal{L}[e^{-t}\,\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs}\,\mathcal{L}[e^{-t}\,\sin(t)].$$

Since for $c \geqslant 0$ holds $e^{-cs}\,\mathcal{L}[f](s) = \mathcal{L}[u(t-c)\,f(t-c)]$, we conclude that

$$y_\delta(t) = u(t-c)\,e^{-(t-c)}\,\sin(t-c).$$

◁

Example 4.4.6: Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: The source is a generalized function, so we need to solve this problem using the Lapace transform. So we compute the Laplace transform of the differential equation,

$$\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)] \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

where in the second equation we have already introduced the initial conditions. We arrive to the equation

$$\mathcal{L}[y] = \frac{s}{(s^2-1)} - 20\,e^{-3s}\,\frac{1}{(s^2-1)} = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\,\sinh(t-3)],$$

which leads to the solution

$$y(t) = \cosh(t) - 20\,u(t-3)\,\sinh(t-3).$$

◁

Example 4.4.7: Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: We again Laplace transform both sides of the differential equation,

$$\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)] \quad \Rightarrow \quad (s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s},$$

where in the second equation above we have introduced the initial conditions. Then,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)} \\
&= \frac{e^{-\pi s}}{2}\,\frac{2}{(s^2+4)} - \frac{e^{-2\pi s}}{2}\,\frac{2}{(s^2+4)} \\
&= \frac{1}{2}\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].
\end{aligned}$$

The last equation can be rewritten as follows,

$$y(t) = \frac{1}{2}\,u(t-\pi)\,\sin\big[2(t-\pi)\big] - \frac{1}{2}\,u(t-2\pi)\,\sin\big[2(t-2\pi)\big],$$

which leads to the conclusion that

$$y(t) = \frac{1}{2}\,\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t).$$

◁

4.4.5. **Comments on Generalized Sources.** We have used the Laplace transform to solve differential equations with the Dirac delta as a source function. It may be convenient to understand a bit more clearly what we have done, since the Dirac delta is not an ordinary function but a generalized function defined by a limit. Consider the following example.

Example 4.4.8: Find the impulse response function at $t = c > 0$ of the linear operator

$$L(y) = y'.$$

Solution: We need to solve the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0.$$

In other words, we need to find a primitive of the Dirac delta. However, Dirac's delta is not even a function. Anyway, let us compute the Laplace transform of the equation, as we did in the previous examples,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\delta(t-c)] \quad \Rightarrow \quad s\,\mathcal{L}[y(t)] - y(0) = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y(t)] = \frac{e^{-cs}}{s}.$$

But we know that

$$\frac{e^{-cs}}{s} = \mathcal{L}[u(t-c)] \quad \Rightarrow \quad \mathcal{L}[y(t)] = \mathcal{L}[u(t-c)] \quad \Rightarrow \quad y(t) = u(t-c).$$

$\triangleleft$

Looking at the differential equation $y'(t) = \delta(t-c)$ and at the solution $y(t) = u(t-c)$ one could like to write them together as

$$u'(t-c) = \delta(t-c). \tag{4.4.3}$$

But this is not correct, because the step function is a discontinuous function at $t = c$, hence not differentiable. What we have done is something different. We have found a sequence of functions u_n with the properties,

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

and we have called $y(t) = u(t-c)$. This is what we actually do when we solve a differential equation with a source defined as a limit of a sequence of functions, such as the Dirac delta. The Laplace transform method used on differential equations with generalized sources allows us to solve these equations without the need to write any sequence, which are hidden in the definitions of the Laplace transform of generalized functions. Let us solve the problem in the Example 4.4.8 one more time, but this time let us show where all the sequences actually are.

Example 4.4.9: Find the solution to the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0, \qquad c > 0, \tag{4.4.4}$$

Solution: Recall that the Dirac delta is defined as a limit of a sequence of bump functions,

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c), \qquad \delta_n(t-c) = n\Big[u(t-c) - u\Big(t-c-\frac{1}{n}\Big)\Big], \qquad n = 1, 2, \cdots.$$

The problem we are actually solving involves a sequence and a limit,

$$y'(t) = \lim_{n\to\infty} \delta_n(t-c), \qquad y(0) = 0.$$

We start computing the Laplace transform of the differential equation,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\lim_{n\to\infty} \delta_n(t-c)].$$

We have defined the Laplace transform of the limit as the limit of the Laplace transforms,

$$\mathcal{L}[y'(t)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)].$$

If the solution is at least piecewise differentiable, we can use the property

$$\mathcal{L}[y'(t)] = s\,\mathcal{L}[y(t)] - y(0).$$

Assuming that property, and the initial condition $y(0) = 0$, we get

$$\mathcal{L}[y(t)] = \frac{1}{s} \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \quad\Rightarrow\quad \mathcal{L}[y(t)] = \lim_{n\to\infty} \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Introduce now the function $y_n(t) = u_n(t-c)$, given in Eq. (4.4.1), which for each n is the only continuous, piecewise differentiable, solution of the initial value problem

$$y_n'(t) = \delta_n(t-c), \qquad y_n(0) = 0.$$

It is not hard to see that this function u_n satisfies

$$\mathcal{L}[u_n(t)] = \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Therefore, using this formula back in the equation for y we get,

$$\mathcal{L}[y(t)] = \lim_{n\to\infty} \mathcal{L}[u_n(t)].$$

For continuous functions we can interchange the Laplace transform and the limit,

$$\mathcal{L}[y(t)] = \mathcal{L}[\lim_{n\to\infty} u_n(t)].$$

So we get the result,

$$y(t) = \lim_{n\to\infty} u_n(t) \quad\Rightarrow\quad y(t) = u(t-c).$$

We see above that we have found something more than just $y(t) = u(t-c)$. We have found

$$y(t) = \lim_{n\to\infty} u_n(t-c),$$

where the sequence elements u_n are continuous functions with $u_n(0) = 0$ and

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

Finally, derivatives and limits cannot be interchanged for u_n,

$$\lim_{n\to\infty} \big[u_n'(t-c)\big] \neq \big[\lim_{n\to\infty} u_n(t-c)\big]'$$

so it makes no sense to talk about y'. ◁

When the Dirac delta is defined by a sequence of functions, as we did in this section, the calculation needed to find impulse response functions must involve sequence of functions and limits. The Laplace transform method used on generalized functions allows us to hide all the sequences and limits. This is true not only for the derivative operator $L(y) = y'$ but for any second order differential operator with constant coefficients.

Definition 4.4.8. *A* ***solution*** *of the initial value problem with a Dirac's delta source*

$$y'' + a_1\,y' + a_0\,y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \tag{4.4.5}$$

where a_1, a_0, y_0, y_1, and $c \in \mathbb{R}$, are given constants, is a function

$$y(t) = \lim_{n\to\infty} y_n(t),$$

where the functions y_n, with $n \geqslant 1$, are the unique solutions to the initial value problems

$$y_n'' + a_1\,y_n' + a_0\,y_n = \delta_n(t-c), \qquad y_n(0) = y_0, \qquad y_n'(0) = y_1, \tag{4.4.6}$$

and the source δ_n satisfy $\lim_{n\to\infty} \delta_n(t-c) = \delta(t-c)$.

The definition above makes clear what do we mean by a solution to an initial value problem having a generalized function as source, when the generalized function is defined as the limit of a sequence of functions. The following result says that the Laplace transform method used with generalized functions hides all the sequence computations.

Theorem 4.4.9. *The function y is solution of the initial value problem*

$$y'' + a_1\, y' + a_0\, y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \qquad c \geqslant 0,$$

iff its Laplace transform satisfies the equation

$$\big(s^2\,\mathcal{L}[y] - sy_0 - y_1\big) + a_1\,\big(s\,\mathcal{L}[y] - y_0\big) - a_0\,\mathcal{L}[y] = e^{-cs}.$$

This Theorem tells us that to find the solution y to an initial value problem when the source is a Dirac's delta we have to apply the Laplace transform to the equation and perform the same calculations as if the Dirac delta were a function. This is the calculation we did when we computed the impulse response functions.

Proof of Theorem 4.4.9: Compute the Laplace transform on Eq. (4.4.6),

$$\mathcal{L}[y_n''] + a_1\,\mathcal{L}[y_n'] + a_0\,\mathcal{L}[y_n] = \mathcal{L}[\delta_n(t-c)].$$

Recall the relations between the Laplace transform and derivatives and use the initial conditions,

$$\mathcal{L}[y_n''] = s^2\,\mathcal{L}[y_n] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y_n] - y_0,$$

and use these relation in the differential equation,

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[\delta_n(t-c)],$$

Since δ_n satisfies that $\lim_{n\to\infty}\delta_n(t-c) = \delta(t-c)$, an argument like the one in the proof of Theorem 4.4.5 says that for $c \geqslant 0$ holds

$$\mathcal{L}[\delta_n(t-c)] = \mathcal{L}[\delta(t-c)] \quad \Rightarrow \quad \lim_{n\to\infty}\mathcal{L}[\delta_n(t-c)] = e^{-cs}.$$

Then

$$(s^2 + a_1 s + a_0)\,\lim_{n\to\infty}\mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = e^{-cs}.$$

Interchanging limits and Laplace transforms we get

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = e^{-cs},$$

which is equivalent to

$$\big(s^2\,\mathcal{L}[y] - sy_0 - y_1\big) + a_1\,\big(s\,\mathcal{L}[y] - y_0\big) - a_0\,\mathcal{L}[y] = e^{-cs}.$$

This establishes the Theorem. □

4.4.6. **Exercises.**

4.4.1.- .

4.4.2.- .