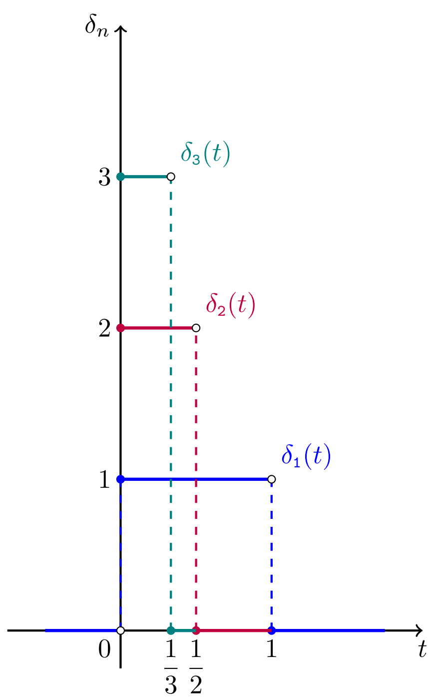
Chapter 4. The Laplace Transform Method

The Laplace Transform is a transformation, meaning that it changes a function into a new function. Actually, it is a linear transformation, because it converts a linear combination of functions into a linear combination of the transformed functions. Even more interesting, the Laplace Transform converts derivatives into multiplications. These two properties make the Laplace Transform very useful to solve linear differential equations with constant coefficients. The Laplace Transform converts such differential equation for an unknown function into an algebraic equation for the transformed function. Usually it is easy to solve the algebraic equation for the transformed function. Then one converts the transformed function back into the original function. This function is the solution of the differential equation.

Solving a differential equation using a Laplace Transform is radically different from all the methods we have used so far. This method, as we will use it here, is relatively new. The Laplace Transform we define here was first used in 1910, but its use grew rapidly after 1920, specially to solve differential equations. Transformations like the Laplace Transform were known much earlier. Pierre Simon de Laplace used a similar transformation in his studies of probability theory, published in 1812, but analogous transformations were used even earlier by Euler around 1737.



4.1. Introduction to the Laplace Transform

The Laplace transform is a transformation—it changes a function into another function. This transformation is an integral transformation—the original function is multiplied by an exponential and integrated on an appropriate region. Such an integral transformation is the answer to very interesting questions: Is it possible to transform a differential equation into an algebraic equation? Is it possible to transform a derivative of a function into a multiplication? The answer to both questions is yes, for example with a Laplace transform.

This is how it works. You start with a derivative of a function, $y'(t)$, then you multiply it by any function, we choose an exponential e^{-st}, and then you integrate on t, so we get

$$y'(t) \to \int e^{-st}\, y'(t)\, dt,$$

which is a transformation, an integral transformation. And now, because we have an integration above, we can integrate by parts—this is the big idea,

$$y'(t) \to \int e^{-st}\, y'(t)\, dt = e^{-st}\, y(t) + s \int e^{-st}\, y(t)\, dt.$$

So we have transformed the derivative we started with into a multiplication by this constant s from the exponential. The idea in this calculation actually works to solve differential equations and motivates us to define the integral transformation $y(t) \to \tilde{Y}(s)$ as follows,

$$y(t) \to \tilde{Y}(s) = \int e^{-st}\, y(t)\, dt.$$

The Laplace transform is a transformation similar to the one above, where we choose some appropriate integration limits—which are very convenient to solve initial value problems.

We dedicate this section to introduce the precise definition of the Laplace transform and how is used to solve differential equations. In the following sections we will see that this method can be used to solve linear constant coefficients differential equation with very general sources, including Dirac's delta generalized functions.

4.1.1. **Oveview of the Method.**

The Laplace transform changes a function into another function. For example, we will show later on that the Laplace transform changes

$$f(x) = \sin(ax) \quad \text{into} \quad F(x) = \frac{a}{x^2 + a^2}.$$

We will follow the notation used in the literature and we use t for the variable of the original function f, while we use s for the variable of the transformed function F. Using this notation, the Laplace transform changes

$$f(t) = \sin(at) \quad \text{into} \quad F(s) = \frac{a}{s^2 + a^2}.$$

We will show that the Laplace transform is a linear transformation and it transforms derivatives into multiplication. Because of these properties we will use the Laplace transform to solve linear differential equations.

We Laplace transform the original differential equation. Because the the properties above, the result will be an algebraic equation for the transformed function. Algebraic equations are simple to solve, so we solve the algebraic equation. Then we Laplace transform back the solution. We summarize these steps as follows,

$$\mathcal{L}\begin{bmatrix}\text{differential} \\ \text{eq. for } y.\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic} \\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the} \\ \text{algebraic} \\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back} \\ \text{to obtain } y. \\ \text{(Use the table.)}\end{matrix}$$

4.1.2. **The Laplace Transform.** The Laplace transform is a transformation, meaning that it converts a function into a new function. We have seen transformations earlier in these notes. In Chapter **??** we used the transformation

$$L[y(t)] = y''(t) + a_1\, y'(t) + a_0\, y(t),$$

so that a second order linear differential equation with source f could be written as $L[y] = f$. There are simpler transformations, for example the differentiation operation itself,

$$D[f(t)] = f'(t).$$

Not all transformations involve differentiation. There are integral transformations, for example integration itself,

$$I[f(t)] = \int_0^x f(t)\, dt.$$

Of particular importance in many applications are integral transformations of the form

$$T[f(t)] = \int_a^b K(s,t)\, f(t)\, dt,$$

where K is a fixed function of two variables, called the *kernel* of the transformation, and a, b are real numbers or $\pm\infty$. The Laplace transform is a transfomation of this type, where the kernel is $K(s,t) = e^{-st}$, the constant $a = 0$, and $b = \infty$.

Definition 4.1.1. *The* ***Laplace transform*** *of a function f defined on $D_f = (0, \infty)$ is*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{4.1.1}$$

defined for all $s \in D_F \subset \mathbb{R}$ where the integral converges.

In these note we use an alternative notation for the Laplace transform that emphasizes that the Laplace transform is a transformation: $\mathcal{L}[f] = F$, that is

$$\mathcal{L}[\quad] = \int_0^\infty e^{-st}\, (\quad)\, dt.$$

So, the Laplace transform will be denoted as either $\mathcal{L}[f]$ or F, depending whether we want to emphasize the transformation itself or the result of the transformation. We will also use the notation $\mathcal{L}[f(t)]$, or $\mathcal{L}[f](s)$, or $\mathcal{L}[f(t)](s)$, whenever the independent variables t and s are relevant in any particular context.

The Laplace transform is an improper integral—an integral on an unbounded domain. Improper integrals are defined as a limit of definite integrals,

$$\int_{t_0}^\infty g(t)\, dt = \lim_{N\to\infty} \int_{t_0}^N g(t)\, dt.$$

An improper integral *converges* iff the limit exists, otherwise the integral *diverges*.

Now we are ready to compute our first Laplace transform.

Example 4.1.1: Compute the Laplace transform of the function $f(t) = 1$, that is, $\mathcal{L}[1]$.

Solution: Following the definition,

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt = \lim_{N\to\infty} \int_0^N e^{-st}\, dt.$$

The definite integral above is simple to compute, but it depends on the values of s. For $s = 0$ we get

$$\lim_{N\to\infty} \int_0^N dt = \lim_{n\to\infty} N = \infty.$$

So, the improper integral diverges for $s = 0$. For $s \neq 0$ we get

$$\lim_{N\to\infty} \int_0^N e^{-st}\,dt = \lim_{N\to\infty} -\frac{1}{s}\, e^{-st}\Big|_0^N = \lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1).$$

For $s < 0$ we have $s = -|s|$, hence

$$\lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1) = \lim_{N\to\infty} -\frac{1}{s}\,(e^{|s|N} - 1) = -\infty.$$

So, the improper integral diverges for $s < 0$. In the case that $s > 0$ we get

$$\lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1) = \frac{1}{s}.$$

If we put all these result together we get

$$\mathcal{L}[1] = \frac{1}{s}, \qquad s > 0.$$

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EXAMPLE 4.1.2: Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

SOLUTION: We start with the definition of the Laplace transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt.$$

In the case $s = a$ we get

$$\mathcal{L}[e^{at}] = \int_0^\infty 1\,dt = \infty,$$

so the improper integral diverges. In the case $s \neq a$ we get

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt, \qquad s \neq a,\\
&= \lim_{N\to\infty} \Big[\frac{(-1)}{(s-a)}\, e^{-(s-a)t}\Big|_0^N\Big]\\
&= \lim_{N\to\infty} \Big[\frac{(-1)}{(s-a)}\,(e^{-(s-a)N} - 1)\Big].
\end{aligned}$$

Now we have to remaining cases. The first case is:

$$s - a < 0 \quad \Rightarrow \quad -(s-a) = |s-a| > 0 \quad \Rightarrow \quad \lim_{N\to\infty} e^{-(s-a)N} = \infty,$$

so the integral diverges for $s < a$. The other case is:

$$s - a > 0 \quad \Rightarrow \quad -(s-a) = -|s-a| < 0 \quad \Rightarrow \quad \lim_{N\to\infty} e^{-(s-a)N} = 0,$$

so the integral converges only for $s > a$ and the Laplace transform is given by

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)}, \qquad s > a.$$

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EXAMPLE 4.1.3: Compute $\mathcal{L}[te^{at}]$, where $a \in \mathbb{R}$.

SOLUTION: In this case the calculation is more complicated than above, since we need to integrate by parts. We start with the definition of the Laplace transform,

$$\mathcal{L}[te^{at}] = \int_0^\infty e^{-st}te^{at}\,dt = \lim_{N\to\infty} \int_0^N te^{-(s-a)t}\,dt.$$

This improper integral diverges for $s = a$, so $\mathcal{L}[te^{at}]$ is not defined for $s = a$. From now on we consider only the case $s \neq a$. In this case we can integrate by parts,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N + \frac{1}{s-a} \int_0^N e^{-(s-a)t}\, dt \Big],$$

that is,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N - \frac{1}{(s-a)^2} e^{-(s-a)t}\Big|_0^N \Big]. \tag{4.1.2}$$

In the case that $s < a$ the first term above diverges,

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = \lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{|s-a|N} = \infty,$$

therefore $\mathcal{L}[te^{at}]$ is not defined for $s < a$. In the case $s > a$ the first term on the right hand side in (4.1.2) vanishes, since

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)} t\, e^{-(s-a)t}\big|_{t=0} = 0.$$

Regarding the other term, and recalling that $s > a$,

$$\lim_{N\to\infty} -\frac{1}{(s-a)^2} e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)^2} e^{-(s-a)t}\big|_{t=0} = \frac{1}{(s-a)^2}.$$

Therefore, we conclude that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}, \qquad s > a.$$

◁

Example 4.1.4: Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \int_0^\infty e^{-st} \sin(at)\, dt = \lim_{N\to\infty} \int_0^N e^{-st} \sin(at)\, dt$$

The definite integral above can be computed integrating by parts twice,

$$\int_0^N e^{-st} \sin(at)\, dt = -\frac{1}{s} \big[e^{-st} \sin(at)\big]\Big|_0^N - \frac{a}{s^2} \big[e^{-st} \cos(at)\big]\Big|_0^N - \frac{a^2}{s^2} \int_0^N e^{-st} \sin(at)\, dt,$$

which implies that

$$\Big(1 + \frac{a^2}{s^2}\Big) \int_0^N e^{-st} \sin(at)\, dt = -\frac{1}{s} \big[e^{-st} \sin(at)\big]\Big|_0^N - \frac{a}{s^2} \big[e^{-st} \cos(at)\big]\Big|_0^N.$$

then we get

$$\int_0^N e^{-st} \sin(at)\, dt = \frac{s^2}{(s^2+a^2)} \Big[-\frac{1}{s} \big[e^{-st} \sin(at)\big]\Big|_0^N - \frac{a}{s^2} \big[e^{-st} \cos(at)\big]\Big|_0^N \Big].$$

and finally we get

$$\int_0^N e^{-st} \sin(at)\, dt = \frac{s^2}{(s^2+a^2)} \Big[-\frac{1}{s} \big[e^{-sN} \sin(aN) - 0\big] - \frac{a}{s^2} \big[e^{-sN} \cos(aN) - 1\big] \Big].$$

One can check that the limit $N \to \infty$ on the right hand side above does not exist for $s \leqslant 0$, so $\mathcal{L}[\sin(at)]$ does not exist for $s \leqslant 0$. In the case $s > 0$ it is not difficult to see that

$$\int_0^\infty e^{-st} \sin(at)\, dt = \Big(\frac{s^2}{s^2+a^2}\Big) \Big[\frac{1}{s}\,(0-0) - \frac{a}{s^2}\,(0-1)\Big]$$

so we obtain the final result

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

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In Table 1 we present a short list of Laplace transforms. They can be computed in the same way we computed the the Laplace transforms in the examples above.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \|a\|$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \|a\|$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$

Table 1. List of a few Laplace transforms.

4.1.3. **Main Properties.** Since we are more or less confident on how to compute a Laplace transform, we can start asking deeper questions. For example, what type of functions have a Laplace transform? It turns out that a large class of functions, those that are piecewise continuous on $[0, \infty)$ and bounded by an exponential. This last property is particularly important and we give it a name.

Definition 4.1.2. *A function f defined on $[0, \infty)$ is of **exponential order** s_0, where s_0 is any real number, iff there exist positive constants k, T such that*

$$|f(t)| \leqslant k\, e^{s_0 t} \quad \text{for all} \quad t > T. \tag{4.1.3}$$

Remarks:

(a) When the precise value of the constant s_0 is not important we will say that f is of exponential order.

(b) An example of a function that is not of exponential order is $f(t) = e^{t^2}$.

This definition helps to describe a set of functions having Laplace transform. Piecewise continuous functions on $[0, \infty)$ of exponential order have Laplace transforms.

Theorem 4.1.3 (Convergence of LT)**.** *If a function f defined on $[0, \infty)$ is piecewise continuous and of exponential order s_0, then the $\mathcal{L}[f]$ exists for all $s > s_0$ and there exists a positive constant k such that*

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s - s_0}, \qquad s > s_0.$$

Proof of Theorem 4.1.3: From the definition of the Laplace transform we know that

$$\mathcal{L}[f] = \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt.$$

The definite integral on the interval $[0, N]$ exists for every $N > 0$ since f is piecewise continuous on that interval, no matter how large N is. We only need to check whether the integral converges as $N \to \infty$. This is the case for functions of exponential order, because

$$\Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant \int_0^N e^{-st} |f(t)|\, dt \leqslant \int_0^N e^{-st} k e^{s_0 t}\, dt = k \int_0^N e^{-(s-s_0)t}\, dt.$$

Therefore, for $s > s_0$ we can take the limit as $N \to \infty$,

$$\big|\mathcal{L}[f]\big| \leqslant \lim_{N\to\infty} \Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant k\, \mathcal{L}[e^{s_0 t}] = \frac{k}{(s - s_0)}.$$

Therefore, the comparison test for improper integrals implies that the Laplace transform $\mathcal{L}[f]$ exists at least for $s > s_0$, and it also holds that

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s - s_0}, \qquad s > s_0.$$

This establishes the Theorem. □

The next result says that the Laplace transform is a linear transformation. This means that the Laplace transform of a linear combination of functions is the linear combination of their Laplace transforms.

Theorem 4.1.4 (Linearity)**.** *If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all a, $b \in \mathbb{R}$ holds*

$$\mathcal{L}[af + bg] = a\, \mathcal{L}[f] + b\, \mathcal{L}[g].$$

Proof of Theorem 4.1.4: Since integration is a linear operation, so is the Laplace transform, as this calculation shows,

$$\begin{aligned}\mathcal{L}[af+bg] &= \int_0^\infty e^{-st}\big[af(t)+bg(t)\big]\,dt\\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt\\ &= a\,\mathcal{L}[f]+b\,\mathcal{L}[g].\end{aligned}$$

This establishes the Theorem. □

Example 4.1.5: Compute $\mathcal{L}[3t^2+5\cos(4t)]$.

Solution: From the Theorem above and the Laplace transform in Table ?? we know that

$$\begin{aligned}\mathcal{L}[3t^2+5\cos(4t)] &= 3\,\mathcal{L}[t^2]+5\,\mathcal{L}[\cos(4t)]\\ &= 3\Big(\frac{2}{s^3}\Big)+5\Big(\frac{s}{s^2+4^2}\Big), \qquad s>0\\ &= \frac{6}{s^3}+\frac{5s}{s^2+4^2}.\end{aligned}$$

Therefore,

$$\mathcal{L}[3t^2+5\cos(4t)] = \frac{5s^4+6s^2+96}{s^3(s^2+16)}, \qquad s>0.$$

◁

The Laplace transform can be used to solve differential equations. The Laplace transform converts a differential equation into an algebraic equation. This is so because the Laplace transform converts derivatives into multiplications. Here is the precise result.

Theorem 4.1.5 (Derivative into Multiplication)**.** *If a function f is continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f']$ exists for $s>s_0$ and*

$$\mathcal{L}[f'] = s\,\mathcal{L}[f]-f(0), \qquad s>s_0. \tag{4.1.4}$$

Proof of Theorem 4.1.5: The main calculation in this proof is to compute

$$\mathcal{L}[f'] = \lim_{N\to\infty}\int_0^N e^{-st}\,f'(t)\,dt.$$

We start computing the definite integral above. Since f' is continuous on $[0,\infty)$, that definite integral exists for all positive N, and we can integrate by parts,

$$\begin{aligned}\int_0^N e^{-st}f'(t)\,dt &= \Big[\Big(e^{-st}f(t)\Big)\Big|_0^N - \int_0^N (-s)e^{-st}f(t)\,dt\Big]\\ &= e^{-sN}f(N)-f(0)+s\int_0^N e^{-st}f(t)\,dt.\end{aligned}$$

We now compute the limit of this expression above as $N\to\infty$. Since f is continuous on $[0,\infty)$ of exponential order s_0, we know that

$$\lim_{N\to\infty}\int_0^N e^{-st}f(t)\,dt = \mathcal{L}[f], \qquad s>s_0.$$

Let us use one more time that f is of exponential order s_0. This means that there exist positive constants k and T such that $|f(t)|\leqslant k\,e^{s_0t}$, for $t>T$. Therefore,

$$\lim_{N\to\infty} e^{-sN}f(N) \leqslant \lim_{N\to\infty} k\,e^{-sN}e^{s_0N} = \lim_{N\to\infty} k\,e^{-(s-s_0)N} = 0, \qquad s>s_0.$$

These two results together imply that $\mathcal{L}[f']$ exists and holds

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0.$$

This establishes the Theorem. ☐

Example 4.1.6: Verify the result in Theorem 4.1.5 for the function $f(t) = \cos(bt)$.

Solution: We need to compute the left hand side and the right hand side of Eq. (4.1.4) and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f'] = \mathcal{L}[-b\sin(bt)] = -b\,\mathcal{L}[\sin(bt)] = -b\,\frac{b}{s^2+b^2} \quad \Rightarrow \quad \mathcal{L}[f'] = -\frac{b^2}{s^2+b^2}.$$

We now compute the right hand side,

$$s\,\mathcal{L}[f] - f(0) = s\,\mathcal{L}[\cos(bt)] - 1 = s\,\frac{s}{s^2+b^2} - 1 = \frac{s^2 - s^2 - b^2}{s^2+b^2},$$

so we get

$$s\,\mathcal{L}[f] - f(0) = -\frac{b^2}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$. ◁

It is not difficult to generalize Theorem 4.1.5 to higher order derivatives.

Theorem 4.1.6 (Higher Derivatives into Multiplication)**.** *If a function f is n-times continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f''], \cdots, \mathcal{L}[f^{(n)}]$ exist for $s > s_0$ and*

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) \tag{4.1.5}$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = s^n\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0). \tag{4.1.6}$$

Proof of Theorem 4.1.6: We need to use Eq. (4.1.4) n times. We start with the Laplace transform of a second derivative,

$$\begin{aligned}\mathcal{L}[f''] &= \mathcal{L}[(f')'] \\ &= s\,\mathcal{L}[f'] - f'(0) \\ &= s\big(s\,\mathcal{L}[f] - f(0)\big) - f'(0) \\ &= s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).\end{aligned}$$

The formula for the Laplace transform of an nth derivative is computed by induction on n. We assume that the formula is true for $n-1$,

$$\mathcal{L}[f^{(n-1)}] = s^{(n-1)}\,\mathcal{L}[f] - s^{(n-2)}\,f(0) - \cdots - f^{(n-2)}(0).$$

Since $\mathcal{L}[f^{(n)}] = \mathcal{L}[(f')^{(n-1)}]$, the formula above on f' gives

$$\begin{aligned}\mathcal{L}[(f')^{(n-1)}] &= s^{(n-1)}\,\mathcal{L}[f'] - s^{(n-2)}\,f'(0) - \cdots - (f')^{(n-2)}(0) \\ &= s^{(n-1)}\,\big(s\,\mathcal{L}[f] - f(0)\big) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0) \\ &= s^{(n)}\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$

This establishes the Theorem. ☐

Example 4.1.7: Verify Theorem 4.1.6 for f'', where $f(t) = \cos(bt)$.

Solution: We need to compute the left hand side and the right hand side in the first equation in Theorem (4.1.6), and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f''] = \mathcal{L}[-b^2 \cos(bt)] = -b^2\, \mathcal{L}[\cos(bt)] = -b^2\, \frac{s}{s^2+b^2} \quad \Rightarrow \quad \mathcal{L}[f''] = -\frac{b^2 s}{s^2+b^2}.$$

We now compute the right hand side,

$$s^2\, \mathcal{L}[f] - s\, f(0) - f'(0) = s^2\, \mathcal{L}[\cos(bt)] - s - 0 = s^2\, \frac{s}{s^2+b^2} - s = \frac{s^3 - s^3 - b^2 s}{s^2+b^2},$$

so we get

$$s^2\, \mathcal{L}[f] - s\, f(0) - f'(0) = -\frac{b^2 s}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f''] = s^2\, \mathcal{L}[f] - s\, f(0) - f'(0)$. ◁

4.1.4. **Solving Differential Equations.** The Laplace transform can be used to solve differential equations. We Laplace transform the whole equation, which converts the differential equation for y into an algebraic equation for $\mathcal{L}[y]$. We solve the Algebraic equation and we transform back.

$$\mathcal{L}\begin{bmatrix}\text{differential} \\ \text{eq. for } y.\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic} \\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the} \\ \text{algebraic} \\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back} \\ \text{to obtain } y. \\ \text{(Use the table.)}\end{matrix}$$

Example 4.1.8: Use the Laplace transform to find y solution of

$$y'' + 9\, y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Remark: Notice we already know what the solution of this problem is. Following § **??** we need to find the roots of

$$p(r) = r^2 + 9 \quad \Rightarrow \quad r_{\pm} = \pm 3\, i,$$

and then we get the general solution

$$y(t) = c_+ \cos(3t) + c_- \sin(3t).$$

Then the initial condition will say that

$$y(t) = y_0\, \cos(3t) + \frac{y_1}{3}\, \sin(3t).$$

We now solve this problem using the Laplace transform method.

Solution: We now use the Laplace transform method:

$$\mathcal{L}[y'' + 9y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear transformation,

$$\mathcal{L}[y''] + 9\, \mathcal{L}[y] = 0.$$

But the Laplace transform converts derivatives into multiplications,

$$s^2\, \mathcal{L}[y] - s\, y(0) - y'(0) + 9\, \mathcal{L}[y] = 0.$$

This is an algebraic equation for $\mathcal{L}[y]$. It can be solved by rearranging terms and using the initial condition,

$$(s^2+9)\, \mathcal{L}[y] = s\, y_0 + y_1 \quad \Rightarrow \quad \mathcal{L}[y] = y_0\, \frac{s}{(s^2+9)} + y_1\, \frac{1}{(s^2+9)}.$$

But from the Laplace transform table we see that

$$\mathcal{L}[\cos(3t)] = \frac{s}{s^2 + 3^2}, \qquad \mathcal{L}[\sin(3t)] = \frac{3}{s^2 + 3^2},$$

therefore,

$$\mathcal{L}[y] = y_0\,\mathcal{L}[\cos(3t)] + y_1\,\frac{1}{3}\,\mathcal{L}[\sin(3t)].$$

Once again, the Laplace transform is a linear transformation,

$$\mathcal{L}[y] = \mathcal{L}\Big[y_0\,\cos(3t) + \frac{y_1}{3}\,\sin(3t)\Big].$$

We obtain that

$$y(t) = y_0\,\cos(3t) + \frac{y_1}{3}\,\sin(3t).$$

$\triangleleft$

4.1.5. **Exercises.**

4.1.1.- .

4.1.2.- .