Chapter 1. Modeling with Ordinary Differential Equations

1.1. Population Models: Exponential Growth, Logistic Equation, Predator-Prey

Mathematical models attempt to describe various situations in physics, engineering, ecology, biology, etc. and ultimately use these descriptions to answer a variety of questions. Ordinary differential equations often study systems that evolve over time, but can depend on other variables/parameters as well. The goal of a mathematical model is not to produce a perfect copy of the real-life situation, but rather to capture the essential features that govern the behavior of the system. What is *essential* is often a judgment call and will depend on the questions we are trying to answer using the model. The modeling approach will greatly depend on the situation (whether you use deterministic or stochastic models, discrete or continuous models, ...), but the following steps are central to creating a mathematical model in almost any setting.

Step 1. Clearly state all the assumptions.

Step 2. Describe all the **independent variables**, **dependent variables**, and **parameters** to be used in the model.

Step 3. Use your assumptions to derive equations relating the variables and parameters.

Step 4. Analyze the predictions of the model - do they make physical sense, do they agree with your data? If not, you might need to revise your assumptions, going back to Step 1.

The above steps probably seem very abstract at this point, but we will see how they are applied to particular modeling scenarios in the next section.

1.2. Population Models: Exponential Growth, Logistic Equation, Predator-Prey

1.2.1. **Exponential Growth.** As a first step, let us model population growth under the assumption of unlimited resources. The approach we are going to take here could apply to a population of bacteria in a large Petri dish, growth of mold on bread, population of rabbits in a forest, human population, etc.

The main assumption we make is that given unlimited resources, *the rate of growth of the population is proportional to the size of the population.* I.e., the more rabbits we have, the more rabbits will be born.

Now, let us clearly denote all our variables and parameters.

- Let t denote time (independent variable),
- let $P(t)$ denote the population (e.g. number of rabbits) at time t (dependent variable),
- let k denote the constant of proportionality between the growth rate and the population size (growth-rate coefficient).

Recall that the rate of growth of $P(t)$ is given by $\dfrac{dP}{dt}$. Given our assumption that the rate of growth of the population P is proportional to $P(t)$ with constant of proportionality k, we arrive at

$$\frac{dP}{dt} = kP. \tag{1.2.1}$$

You might have already encountered this type of equation in Calculus II. It is one of the simplest examples of **separable equations**, which will be studied in detail in Section **??**. We will postpone the derivation of the analytic solution to the next chapter. For now, to analyze the predictions of the model, let us look at the graphs of several functions satisfying Equation 1.2.1 with different initial conditions (different values of $P(0)$).

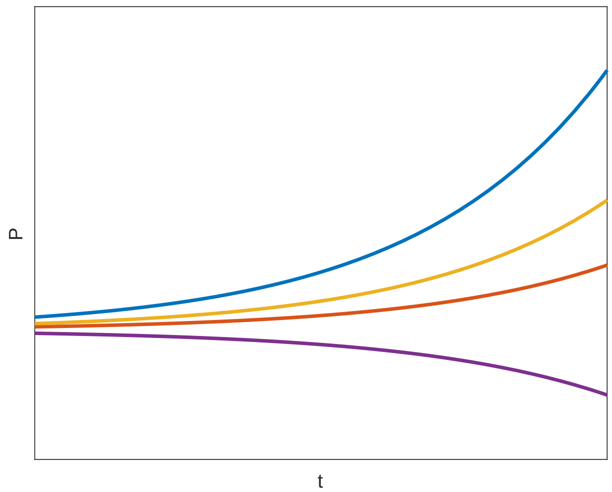

FIGURE 1. Solutions of the exponential growth/decay equation 1.2.1 for different values of the initial condition, fixed value of k.

Now, let us consider how the solutions depend on the parameter k. Before looking at the graph below, try to predict what will happen to the graph if we increase k and what will happen if we decrease k. Could we choose k to be negative? Why or why not?

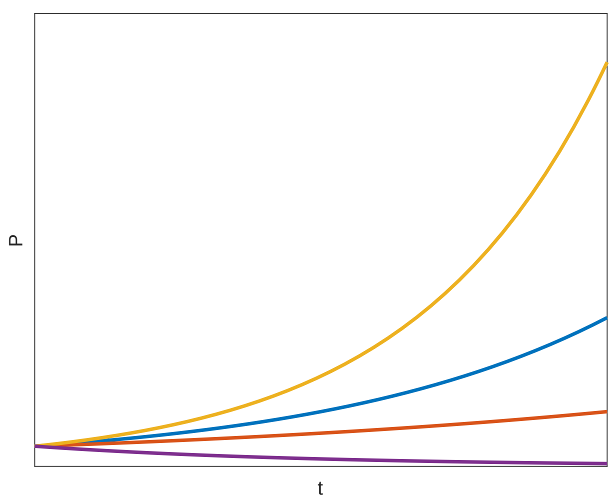

FIGURE 2. Solutions of the exponential growth/decay equation 1.2.1 for different values of k, fixed initial condition.

Under what conditions does this make physical sense? Is it realistic for the number of rabbits to grow to infinity as t grows? This model, while very useful to predict the behavior of the population in the short term, and while the population has not felt that resources are not really unlimited, clearly has some limitations. For this reason we consider the so called **logistic model** to account for limited resources (in terms of food, space, etc.).

1.2.2. **Logistic Population Model.** In this section we model population growth under limited resources. We assume that if the population is small, the rate of growth of the population is proportional to its size (as in the exponential growth model). However, if the population is too large to be supported by the resources in the environment, we assume the population will decrease. The terminology used by ecologists for the latter case is "The population exceeds the carrying capacity of the environment."

As in the previous model, we denote all our variables and parameters.

- Let t denote time,
- let $P(t)$ denote the population at time t,
- let k denote the growth-rate coefficient for small values of P,
- and finally, let N denote the carrying capacity.

That is, we assume that $P(t)$ will decrease (i.e., $P'(t) < 0$) provided $P(t) > N$ and $P'(t) \approx kP(t)$ if P is relatively small (compared to N). We are looking for the simplest equation that would satisfy the above conditions. Thus, we are looking for an equation of the form

$$\frac{dP}{dt} = kP \cdot (???)$$

We would like (???) to be as simple as possible and such that it is positive for $P < N$ and negative for $P > N$. That is, something similar to $N - P$ would do. However, we also assumed that $P'(t) \approx kP(t)$ if P is small, so we should divide $(N - P)$ by N, arriving at the following model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right). \tag{1.2.2}$$

We refer to 1.2.2 as the **logistic population model** with growth rate k and carrying capacity N. This is again a separable equation, for which we can obtain an analytic solution. In Section **??** we will do *qualitative analysis* to see what the model predicts in the long run (for large values of t).

1.2.3. **Predator-Prey Systems.** In many cases we might be interested in modeling the interaction between two or more species. In Chapter **??** we will study a variety of examples of predator-prey (also referred to as Lotka-Volterra) systems, competing species, symbiosis, etc. In this section, we will briefly discuss modeling the population of one species of predator and one species of prey living in the same environment. Consider, for example, foxes and rabbits living in a forest. We make the following assumptions.

(1) In the absence of foxes, the rabbits grow exponentially (we assume unlimited vegetation).
(2) The death rate of rabbits is proportional to the rate at which foxes and rabbits interact (usually modeled as the product of the two populations).
(3) In the absence of rabbits, the foxes die out at a rate proportional to their numbers (exponential decay).
(4) The birth rate of foxes is proportional to the number of rabbits and number of foxes (the more rabbits there are the more foxes are born and the more foxes there are the more foxes are born).

We will use the following notation:

- As usual, t denotes time.
- Let $F(t)$ denote the number of foxes at time t,
- $R(t)$ - the number of rabbits at time t,
- α - growth rate coefficient of the rabbits,
- β - the death rate coefficient of rabbits due to the fox-rabbit interaction,
- γ - the death rate coefficient of foxes,
- δ - the birth rate coefficient of foxes (the constant of proportionality which measures the benefit to the fox due to the rabit-fox interaction).

Note that all the parameters in the model are positive. Taking the above assumptions into account, we arrive at the following model

$$\begin{aligned}\frac{dR}{dt} &= \alpha R - \beta RF \\ \frac{dF}{dt} &= -\gamma F + \delta RF.\end{aligned} \tag{1.2.3}$$

Even though 1.2.3 seems too simplistic to model a realistic situation, equations of this type are often used in economics, ecology, biology, chemistry, etc. This model was initially proposed by Alfred Lotka in 1910 as a model of autocatalytic chemical reactions. It was later independently developed and successfully used by Vito Volterra to explain the increased number of sharks in the Adriatic Sea following World War I.

Unlike the exponential growth model, 1.2.1, and the logistic model, 1.2.2, for most values of the parameters the predator-prey system has no analytic solution, i.e. we cannot find expressions for $R(t)$ and $F(t)$ in terms of explicit formulas. Nonetheless, we can describe the behavior of the system using qualitative approaches and we can find numerical solutions to the system. In this course we will employ all there approaches (analytic, qualitative, and numerical) to gain a better understanding of the differential equations we are studying.