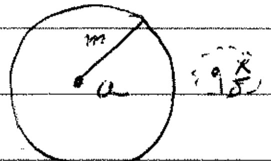
HW 7 - solution key



8.3.2 $n \in \mathbb{N}$, $\underline{a} \in \mathbb{R}^n$, $s, r \in \mathbb{R}$ s.t. $s < r$ and
$V = \{\underline{x} \in \mathbb{R}^n : s < \|\underline{x} - \underline{a}\| < r\}$
$E = \{\underline{x} \in \mathbb{R}^n : s \le \|\underline{x} - \underline{a}\| \le r\}$

Prove that $V$ is open and $E$ is closed.

proof: Note that $V = B_r(\underline{a}) \setminus C_s(\underline{a}) = B_r(\underline{a}) \cap (C_s(\underline{a}))^c$
where $C_s(\underline{a}) = \{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{a}\| \le s\}$.
Similarly, $E = C_r(\underline{a}) \cap (B_s(\underline{a}))^c$.
Since, we've proven that $\forall m > 0$ $B_m(\underline{a})$ is open, then $(B_m(\underline{a}))^c$ is closed.
It suffices to prove $C_m(\underline{a})$ - closed $\forall m$, as then $(C_s(\underline{a}))^c$ is open and $V$ - open as the intersection of 2 open sets. Similarly, $E$ - closed as the intersection of 2 closed sets.
Now, let $m > 0$ and $\underline{a} \in \mathbb{R}^n$. We'll show
$C_m(\underline{a}) = \{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{a}\| \le m\}$ is closed, i.e. we'll show $(C_m(\underline{a}))^c = \{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{a}\| > m\}$ is open.

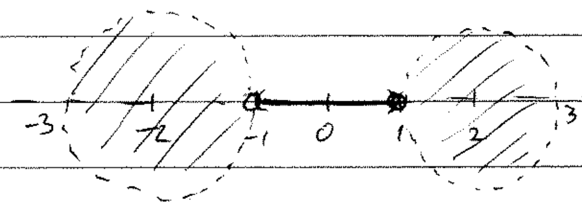



Let $\underline{x} \in (C_m(\underline{a}))^c$ and define
$$\delta = \frac{1}{2}(\|\underline{x} - \underline{a}\| - m) > 0$$

We'll show $B_\delta(\underline{x}) \subseteq (C_m(\underline{a}))^c$, thus demonstrating $\forall \underline{x} \in (C_m(\underline{a}))^c$ $\exists \delta > 0$ s.t. $B_\delta(\underline{x}) \subseteq (C_m(\underline{a}))^c$, i.e. $(C_m(\underline{a}))^c$ is open.
Indeed, Let $\underline{z} \in B_\delta(\underline{x})$, so $\|\underline{x} - \underline{z}\| < \delta$. Then
Now, note that $\|\underline{x} - \underline{a}\| \le \|\underline{x} - \underline{z}\| + \|\underline{z} - \underline{a}\|$.
Thus $\|\underline{z} - \underline{a}\| \ge \|\underline{x} - \underline{a}\| - \|\underline{x} - \underline{z}\| > m$

since $\|\underline{x} - \underline{z}\| < \delta < \|\underline{x} - \underline{a}\| - m$. ∎

**8.3.3 b** $E = B_1(-2,0) \cup B_1(2,0) \cup \{(x,0) : -1 < x < 1\}$



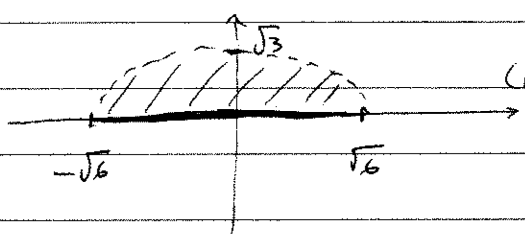

The set $E$ is not connected - it can be separated by the following two sets:

$U = \{(x,y) \in \mathbb{R}^2 : x > 1\}$
$V = \{(x,y) \in \mathbb{R}^2 : x < 1\}$

$U$ and $V$ are open, $U \cap V = \phi$, $U \cap E \neq \phi$, for example, $(2,0) \in U \cap E$, similarly $V \cap E \neq \phi$ - for example, $(-2,0) \in V \cap E$. Finally $E \subseteq U \cup V$, since $U \cup V = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x = 1\}$, and $E$ has no points with x-coordinate 1.

**8.3.4** $E_1 = \{(x,y) : y \geq 0\}$ and $E_2 = \{(x,y) : x^2 + 2y^2 < 6\}$

$U = \{(x,y) : x^2 + 2y^2 < 6 \text{ and } y \geq 0\}$




(b) $U$ is relatively open w.r.t. $E_1$, since $U = E_1 \cap E_2$ and $E_2$ is open.

(c) $U$ is relatively closed with respect to $E_2$, since $U = E_2 \cap E_1$ and $E_1$ is closed.

**8.3.7** a) If $A$ and $B$ are connected and $A \cap B \neq \phi$, prove $A \cup B$ is connected.

*proof:* Assume, by way of contradiction, $A \cup B$ is not connected. Then $\exists\ \mathcal{U}, V$ - open s.t. $\mathcal{U} \cap V = \phi$, $\mathcal{U} \cap (A \cup B) \neq \phi$, $V \cap (A \cup B) \neq \phi$ and $(A \cup B) \subseteq (\mathcal{U} \cup V)$.
We assumed $A \cap B \neq \phi$, thus $\exists\ \underline{x} \in A \cap B \subseteq \mathcal{U} \cup V$, thus $\underline{x} \in \mathcal{U}$ or $\underline{x} \in V$. Without loss of generality, assume $\underline{x} \in \mathcal{U}$.

*Case 1* $A \cap V \neq \phi$, then $\mathcal{U}$ & $V$ separate $A$ (as $\underline{x} \in A \cap \mathcal{U} \neq \phi$).

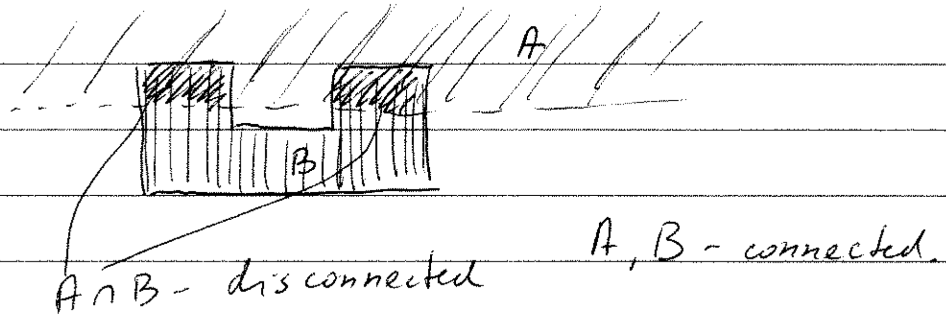
*Case 2* $A \cap V = \phi$. But we assumed $V \cap (A \cup B) \neq \phi$, thus we must have $V \cap B \neq \phi$, we also have $\underline{x} \in B \cap \mathcal{U} \neq \phi$, thus $\mathcal{U}$ & $V$ separate $B$.

In each case we reach a contradiction, since $A, B$ are connected. Thus, we conclude $A \cup B$ must be connected. ∎

(c) If $A$ and $B$ are connected in $\mathbb{R}$ and $A \cap B \neq \phi$, prove $A \cap B$ is connected.

*proof:* $A, B \subseteq \mathbb{R}$ are connected $\Rightarrow$ $A$ & $B$ must be intervals, thus $A \cap B$ is an interval (possibly a point), i.e. is connected.

(d) Ex



**8.3.8 a** Let $V \subseteq \mathbb{R}^n$. Prove that $V$ is open iff $\exists$ a collection of open balls $\{B_\alpha, \alpha \in A\}$ s.t. $V = \bigcup_{\alpha \in A} B_\alpha$.

proof:

($\Leftarrow$) If $V = \bigcup_{\alpha \in A} B_\alpha$, where $B_\alpha$ - open balls, then $V$ - open. ✓

($\Rightarrow$) Assume $V$ - open. Let $\underline{x} \in V$, then $\exists\, \delta_x > 0$ s.t. $B_{\delta_x}(\underline{x}) \subseteq V$. Thus $V \subseteq \bigcup_{\underline{x} \in V} B_{\delta_x}(\underline{x}) \subseteq V$,

i.e. $V = \bigcup_{\underline{x} \in V} B_{\delta_x}(\underline{x})$. ∎

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**8.3.9** Show that if $E$ is closed in $\mathbb{R}^n$ and $\underline{a} \notin E$, then $\inf_{\underline{x} \in E} \|\underline{x} - \underline{a}\| > 0$.

proof: $E$ - closed, so $E^c$ - open and $\underline{a} \in E^c$, thus $\exists\, \delta > 0$ s.t. $B_\delta(\underline{a}) \subseteq E^c$, i.e.

if $\|\underline{x} - \underline{a}\| < \delta$, then $\underline{x} \in E^c$. The contrapositive of this is: if $\underline{x} \in E$, then $\|\underline{x} - \underline{a}\| \geq \delta$. Therefore,

$$\inf_{\underline{x} \in E} \|\underline{x} - \underline{a}\| \geq \delta > 0.$$

∎