8.2.11 - bonus $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ - given.

Set $M_1 := \sup_{\|\vec{x}\|=1} \|T(\vec{x})\|$

$M_2 := \inf \{ C > 0 : \|T(\vec{x})\| \le C\|\vec{x}\| \ \ \forall \vec{x} \in \mathbb{R}^n \}$

a) Prove $M_1 \le \|T\|$

Recall $\|T\| = \sup_{\vec{x} \ne \vec{0}} \dfrac{\|T(\vec{x})\|}{\|\vec{x}\|} \geqslant \sup_{\|\vec{x}\|=1} \dfrac{\|T(\vec{x})\|}{\|\vec{x}\|=1} = M_1$

(Here sup is taken over a larger set.)

b) Prove: if $\vec{x} \ne \vec{0}$, then $\dfrac{\|T(\vec{x})\|}{\|\vec{x}\|} \le M_1$.

$$\frac{\|T(\vec{x})\|}{\|\vec{x}\|} = \left\| \frac{1}{\|\vec{x}\|} T(\vec{x}) \right\| = \left\| T\left(\frac{\vec{x}}{\|\vec{x}\|}\right) \right\| \le M_1.$$

since $\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = 1$

c) Prove $M_1 = M_2 = \|T\|$.

First, we'll show $M_1 = \|T\|$. Indeed, by (a) $M_1 \le \|T\|$. By (b), taking sup over all $\vec{x} \ne \vec{0}$

$$\underline{\underline{\|T\|}} = \sup_{\vec{x} \ne \vec{0}} \frac{\|T(\vec{x})\|}{\|\vec{x}\|} \le \sup_{\vec{x} \ne \vec{0}} M_1 = \underline{\underline{M_1}}$$

Thus $M_1 = \|T\|$.

By Th. 8.17 we know $\|T(\vec{x})\| \le \|T\| \, \|\vec{x}\| \ \forall \vec{x} \in \mathbb{R}^n$

Thus, $M_2 \le \|T\|$.

It remains to show $M_2 \geqslant \|T\|$.

By the Approximation Property of inf, $\exists$ sequence $\{C_n\}_{n=1}^{\infty}$ s.t. $C_n \to M_2$ and

$\forall n \in \mathbb{N}$: $\|T(\vec{x})\| \le C_n \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$, thus

$\|T\| = \sup_{\vec{x} \ne \vec{0}} \dfrac{\|T(\vec{x})\|}{\|x\|} \le C_n$. Take limit as $n \to \infty$

$$\|T\| \le \lim_{n\to\infty} C_n = M_2.$$

8.2.5 b $T(1,1,0) = (e,\pi)$

$T(0,-1,1) = (1,0)$

$T(1,1,-1) = (1,2)$

$T(\vec{e}_1) = T(0,-1,1) + T(1,1,-1) = (2,2)$

$T(\vec{e}_2) = T(1,1,0) - T(\vec{e}_1) = (e-2, \pi-2)$

$T(\vec{e}_3) = T(0,-1,1) + T(\vec{e}_2) = (e-1, \pi-2)$

The matrix B of T is given by

$$B = \begin{bmatrix} 2 & e-2 & e-1 \\ 2 & \pi-2 & \pi-2 \end{bmatrix}.$$

8.2.10 a Find $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^m)$ s.t.

$$\lim_{h\to 0} \frac{\|\vec{f}(x+h) - \vec{f}(x) - T(h)\|}{h} = 0.$$

$f(x) = (x^2, \sin x)$.

$$f(x+h) = ((x+h)^2, \sin(x+h)) = \left(x^2+2xh+h^2, \sin x + (\cos x)h - \frac{(\sin x)h^2}{2} + \dots\right)$$

$$= (x^2, \sin x) + (2x, \cos x)h + \left(1, -\frac{\sin x}{2}\right)h^2 + O(h^3).$$

Set $T(h) = (2x, \cos x)h$, $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$.

Then $\dfrac{\|f(x+h) - f(x) - T(h)\|}{h} = \dfrac{h^{\cancel{2}}}{\cancel{h}} \left\|\left(1, -\dfrac{\sin x}{2}\right)\right\| \xrightarrow[h\to 0]{} 0.$

8.2.4 $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $m, n \in \mathbb{N}$

(a) $T(x, y, z, w) = (0, x+y, x-z, x+y+w)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = T(x, y, z, w).$$

(b) $T(x, y, z) = x - y + z$

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T(x, y, z)$$

(c) $T(x_1, \ldots, x_n) = (x_1 - x_n, x_n - x_1)$

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & -1 \\ -1 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = T(x_1, \ldots, x_n).$$

8.2.5 **a** $T(1,1) = (3, \pi, 0)$ $T(0,1) = (4, 0, 1)$

$\Rightarrow T(1,0) = T(1,1) - T(0,1) = (-1, \pi, -1)$.

The matrix B of T is given by $B = \begin{bmatrix} T\vec{e}_1 & T\vec{e}_2 \\ | & | \end{bmatrix}$

Thus, $B = \begin{bmatrix} -1 & 4 \\ \pi & 0 \\ -1 & 1 \end{bmatrix}$.

HW 6 - Solution Key

8.2.2 a

$P_1 = (1, 0, 0, 0)$ $\qquad P_2 - P_1 = (1, 1, 0, 0)$

$P_2 = (2, 1, 0, 0)$ $\qquad P_3 - P_1 = (-1, 1, 1, 0)$

$P_3 = (0, 1, 1, 0)$ $\qquad P_4 - P_1 = (-1, 4, 0, 1)$

$P_4 = (0, 4, 0, 1)$

$$\Pi_{\vec{n}}(\vec{a}) = \{ \vec{x} \in \mathbb{R}^n : \vec{n} \cdot (\vec{x} - \vec{a}) = 0 \}$$

$\vec{n} = (n_1, n_2, n_3, n_4)$ $\qquad \vec{n} \perp P_i - P_1$, $i = 1, 2, 3$.

$$\begin{cases} n_1 + n_2 = 0 \\ -n_1 + n_2 + n_3 = 0 \\ -n_1 + 4n_2 + n_4 = 0 \end{cases}$$

Choose $n_1 = 1$

Then $n_2 = -1$

$n_3 = 2$

$n_4 = 5$

Eq'n of the hyperplane: $x_1 - x_2 + 2x_3 + 5x_4 = 1$.

---

b Find $\Pi_{\vec{n}}(\vec{a}) \subseteq \mathbb{R}^4$ containing

$\varphi(t) = (t, t, t, 1)$

$\psi(t) = (1, t, 1+t, t)$

4 points on $\Pi_{\vec{n}}(\vec{a})$ are $\varphi(0) = (0, 0, 0, 1)$

$\varphi(1) = (1, 1, 1, 1)$

$\psi(0) = (1, 0, 1, 0)$

$\psi(1) = (1, 1, 2, 1)$

Repeat the method used in (a).

---

c Hyperplane Π parallel to $x_1 + \ldots + x_n = \pi$ and passing through $\vec{a} = (1, 2, \ldots, n)$.

Then $\vec{n} = (1, \ldots, 1)$ is $\perp$ to Π, thus

Π: $\quad \vec{n} \cdot (\vec{x} - \vec{a}) = 0$

$$x_1 + x_2 + \ldots + x_n = \vec{n} \cdot \vec{a} = \frac{n(n+1)}{2}$$