

- I. Be able to state important definitions and theorems.
 - II. Review homework problems.
 - III. Review quizzes and previous exams.
 - IV. Be able to prove short and straightforward theorems.
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Important definitions you should be able to state

1. Upper bound, lower bound, supremum, infimum for a set in \mathbb{R} .
 2. What do we mean by $\lim_{n \rightarrow \infty} s_n = L$, $\lim_{n \rightarrow \infty} s_n = +\infty$, $\lim_{n \rightarrow \infty} s_n = -\infty$?
 3. Increasing, decreasing sequence.
 4. Definition of limsup and liminf.
 5. Cauchy sequence.
 6. Continuity of a function (sequential and $\varepsilon - \delta$ property)
 7. Uniform continuity
 8. Limit of a function as x tends to a along a set S (Def. 20.1)
 9. Two-sided limit, left hand limit, right hand limit ($\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a^+} f(x) = L$, $\lim_{x \rightarrow a^-} f(x) = L$)
 10. Define each of the following limits in terms of quantified implications ($\varepsilon - \delta$ definition, or $M - \delta$, etc.): $\lim_{x \rightarrow a^-} f(x) = +\infty$, $\lim_{x \rightarrow -\infty} f(x) = +\infty$, ...
 11. Radius of convergence, interval of convergence of a power series.
 12. Uniform convergence of a sequence of functions on a given set
 13. Derivative of a function
 14. Increasing/decreasing, strictly increasing/decreasing function
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Important theorems/axioms you should be able to state/prove

1. Proving basic properties of real numbers based on given axioms.
2. Completeness Axiom (state)
3. Proving basic properties of inf and sup (see problems 4.5-4.9, 4.14, 4.16)
4. Proving a given sequence converges using the definition (see Sect. 8)
5. Prove: Convergent sequences are bounded. (see Sect. 9)

6. Prove: If (s_n) and (t_n) converge, for $a, b \in \mathbb{R}$, prove $\lim_{n \rightarrow \infty} (as_n + bt_n) = a \lim_{n \rightarrow \infty} s_n + b \lim_{n \rightarrow \infty} t_n$. (see Sect. 9)
7. Proving a given sequence converges using limit theorems (see Sect. 9)
8. Prove: Given $s_n > 0$ for all $n \in \mathbb{N}$, prove $\lim_{n \rightarrow \infty} s_n = +\infty$ if and only if $\lim_{n \rightarrow \infty} \frac{1}{s_n} = 0$. (see Sect. 9)
9. Be able to use: All bounded monotone sequences converge.
10. Prove: If a sequence is convergent, then it is Cauchy. (Sect 10)
11. State and apply: Comparison, Ratio, Root, Alternating Series, Integral Tests
12. Prove basic properties on how to work with convergent infinite series - add, multiply by a scalar, etc. (See Problem 14.5)
13. Prove: Th. 17. 2 and 20.6 - note the differences and similarities in the proofs.
14. Prove: continuity of a function at a given point using the definition and the $\varepsilon - \delta$ properties (See Examples 1, 2, 3 in Sect. 17 and related HW.)
15. Prove: properties about operations on continuous functions (See Th. 17.4, 17.5)
16. State and apply: Extreme Value Theorem (Th. 18.1), Intermediate Value Theorem (Th. 18.2)
17. State and apply: Fundamental theorems on uniform continuity (Th. 19.2, Th. 19.4, Th. 19.5)
18. Prove: properties of limits of functions (See Th. 20.4, Th. 20.5)
19. State and apply: theorem on radius of convergence of a power series (Th. 23.1, Cor. 12.3)
20. State and apply: theorem on uniform limit of continuous functions (Th. 24.3)
21. State and apply: Weierstrass M-Test (Th. 25.5 and 25.6)
22. Prove: basic properties of derivatives (Th. 28.3(i,ii,iii))
23. State and apply: Rolle's Theorem, Mean Value Theorem
24. Prove: Cor. 29.4, 29.5, 29.7
25. State and apply: L'Hospital's Rule

In addition, these problems might help you review the material.

1. Let A be a nonempty subset of \mathbb{R} that is bounded below, and $B = \{a + 7 : a \in A\}$. Prove that B is also bounded below and $\inf B = 7 + \inf A$.
2. Use the definition of convergence to prove $\lim_{n \rightarrow \infty} \frac{3n^3 + 7n + 1}{n^3 - n - 3} = 3$.
3. Let $a_n = 5 + 3(-1)^n$ for $n \in \mathbb{N}$. Prove (a_n) diverges.
4. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{5}(3a_n + 1)$ for $n \geq 1$. Prove $\lim_{n \rightarrow \infty} a_n$ exists and find the limit.
5. Use the definition to prove that if $\lim_{n \rightarrow \infty} a_n = -\infty$ and $\lim_{n \rightarrow \infty} b_n = 7$, then $\lim_{n \rightarrow \infty} (2a_n + 3b_n) = -\infty$
6. Determine which of the following series converge. Justify your answer.

- (a) $\sum_{n=0}^{\infty} \frac{n^3}{5^n}$
- (b) $\sum_{n=1}^{\infty} \frac{n-5}{2n}$
- (c) $\sum_{n=1}^{\infty} \frac{n-5}{2n^2}$
- (d) $\sum_{n=1}^{\infty} \frac{n+5}{2n^3}$
- (e) $\sum_{n=0}^{\infty} \frac{3}{(-2)^n}$ - calculate the sum.
- (f) $\sum_{n=1}^{\infty} \frac{3(-1)^n}{n}$
- (g) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$

7. Prove that the function $\sqrt{4x+1}$ is continuous at $x_0 = 2$ by verifying the $\varepsilon - \delta$ property.
8. Prove that the function $\sqrt{4x+1}$ is uniformly continuous on $[2, \infty)$ by directly verifying the $\varepsilon - \delta$ property.
9. Prove that the function $f(x) = ((2x-1)^6 + \sqrt{3x^2+16})^5$ is uniformly continuous on $(0, 10)$ using the appropriate theorems.

10. Complete the following statements:

- (a) $\lim_{x \rightarrow a^-} f(x) = +\infty$ if and only if for any _____ there exists _____ such that _____ implies _____.
- (b) $\lim_{x \rightarrow a} f(x) = L$ if and only if for any _____ there exists _____ such that _____ implies _____.
- (c) $\lim_{x \rightarrow -\infty} f(x) = L$ if and only if for any _____ there exists _____ such that _____ implies _____.

11. Prove, using the above problem.

- (a) $\lim_{x \rightarrow 9^-} \frac{7}{9-x} = +\infty$.
- (b) $\lim_{x \rightarrow 10} \frac{7}{9-x} = -7$. (How could we prove this in a simpler way than using the above problem?)
- (c) $\lim_{x \rightarrow \infty} \frac{7x}{9-x} = -7$

12. For each of the following series find the radius of convergence and determine the exact interval of convergence.

- (a) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n$

(b) $\sum_{n=1}^{\infty} \frac{5^n}{n!} x^n$

(c) $\sum_{n=1}^{\infty} \frac{7^n}{n5^n} x^n$

(d) $\sum_{n=1}^{\infty} 5^{-n} x^{2n}$

13. For $x \in \mathbb{R}$, let $f_n(x) = \frac{x}{n^2 + 1}$.

(a) Find $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

(b) Determine whether $f_n \rightarrow f$ uniformly on $[-5, 10]$.

(c) Determine whether $f_n \rightarrow f$ uniformly on $[5, \infty)$.

14. For $x \in [0, \infty)$, let $f_n(x) = \frac{1}{x^n + 1}$.

(a) Find $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

(b) Determine whether $f_n \rightarrow f$ uniformly on $[0, 5]$.

15. Does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ represent a continuous function on $[-10, 10]$? Explain your reasoning.

16. Use the definition of derivative to calculate

(a) $f'(1)$ for $f(x) = \sqrt{3x + 6}$.

(b) $g'(0)$ for $g(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0$ and $g(0) = 0$.

17. Use the Mean Value Theorem to show $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

18. Show $ex \leq e^x$ for all $x \in \mathbb{R}$.

19. Find $\lim_{x \rightarrow 0} (1 + 5x)^{1/x}$.

20. Find $\lim_{x \rightarrow \infty} (e^x + x)^{1/x}$.