**Math 320-001** **Review for Exam 2** **Fall, 2016**

This exam focuses on Sections 10, 11, 12, 14, 15, 17, 18, 19, 20.

I. Be able to state important definitions and theorems.

II. Review homework problems and in-class worksheets.

III. Review quizzes.

IV. Be able to prove short and straightforward theorems.

## Important definitions you should be able to state

1. Definition of limsup and liminf.
2. Cauchy sequence.
3. Convergence of an infinite series
4. Continuity of a function (sequential and $\varepsilon - \delta$ property)
5. Uniform continuity
6. Limit of a function as $x$ tends to $a$ along a set $S$ (Def. 20.1)
7. Two-sided limit, left hand limit, right hand limit ($\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a^+} f(x) = L$, $\lim\limits_{x \to a^-} f(x) = L$)
8. Define each of the following limits in terms of quantified implications ($\varepsilon - \delta$ definition, or $M - \delta$, etc. ): $\lim\limits_{x \to a^-} f(x) = +\infty$, $\lim\limits_{x \to -\infty} f(x) = +\infty$, ...

## Important theorems you should be able to state/prove

1. Prove: If a sequence is convergent, then it is Cauchy. (Sect 10)
2. State and apply: Bolzano-Weierstrass Theorem
3. State and apply: Comparison, Ratio, Root, Alternating Series, Integral Tests
4. Prove basic properties on how to work with convergent infinite series - add, multiply by a scalar, etc. ( See Problem 14.5)
5. Prove: Th. 17. 2 and 20.6 - note the differences and similarities in the proofs.
6. Prove: continuity of a function at a given point using the definition and the $\varepsilon - \delta$ properties (See Examples 1, 2, 3 in Sect. 17 and related HW.)
7. Prove: properties about operations on continuous functions (See Th. 17.4, 17.5)
8. State and apply: Extreme Value Theorem (Th. 18.1), Intermediate Value Theorem (Th. 18.2)
9. State and apply: Fundamental theorems on uniform continuity (Th. 19.2, Th. 19.4, Th. 19.5)
10. Prove: properties of limits of functions (See Th. 20.4, Th. 20.5)

## In addition, these problems might help you review the material.

1. Determine which of the following series converge. Justify your answer.

   (a) $\sum_{n=0}^{\infty} \frac{n^3}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{n-5}{2n}$ (c) $\sum_{n=1}^{\infty} \frac{n-5}{2n^2}$ (d) $\sum_{n=1}^{\infty} \frac{n+5}{2n^3}$ (e) $\sum_{n=0}^{\infty} \frac{3}{(-2)^n}$ - calculate the sum.

   (f) $\sum_{n=1}^{\infty} \frac{3(-1)^n}{n}$ (g) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$.

2. Prove that the function $f$ defined by $f(x) = 3$, if $x \in \mathbb{Q}$ and $f(x) = 5$, if $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at every point $x \in \mathbb{R}$.

3. Prove that the function

$$f(x) = \begin{cases} x^2 - 3, & x > 3 \\ x + 5, & x \leq 3. \end{cases}$$

   is discontinuous at $x = 3$.

4. Prove that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} + 7, & x \neq 0 \\ 7, & x = 0. \end{cases}$$

   is continuous at $x = 0$.

5. Prove that the function $\sqrt{4x+1}$ is continuous at $x_0 = 2$ by verifying the $\varepsilon - \delta$ property.

6. Prove that the function $\sqrt{4x+1}$ is uniformly continuous on $[2, \infty)$ by directly verifying the $\varepsilon - \delta$ property.

7. Prove that the function $f(x) = ((2x-1)^6 + \sqrt{3x^2 + 16})^5$ is uniformly continuous on $(0, 10)$ using the appropriate theorems.

8. Prove that the function $f(x) = \dfrac{5}{x-3}$ is NOT uniformly continuous on $(3, 4)$, using the appropriate theorems.

9. Complete the following statements:

   (a) $\lim_{x \to a^-} f(x) = +\infty$ if and only if for any ____________ there exits ____________ such that ____________ implies ____________.

   (b) $\lim_{x \to a} f(x) = L$ if and only if for any ____________ there exits ____________ such that ____________ implies ____________.

   (c) $\lim_{x \to -\infty} f(x) = L$ if and only if for any ____________ there exits ____________ such that ____________ implies ____________.

10. Prove, using the above problem.

    (a) $\lim_{x \to 9^-} \frac{7}{9-x} = +\infty$.

    (b) $\lim_{x \to 10} \frac{7}{9-x} = -7$. (How could we prove this in a simpler way than using the above problem?)

    (c) $\lim_{x \to \infty} \frac{7x}{9-x} = -7$