Math 310-001 **Review for Final** **Fall, 2014**

I. Review homework problems.

II. Review quizzes.

III. Be able to prove short and straightforward theorems.

IV. Refer to the reviews for Exams I and II, as well as the exams themselves.

Some practice problems to review sections covered in Chapter 6 and Chapter 7

1. Show that the set of all polynomials with a constant coefficient which is divisible by 5 is an ideal in $\mathbb{Z}[x]$. On the other hand, show that the set P of all polynomials with a leading coefficient which is divisible by 5 is NOT an ideal in $\mathbb{Z}[x]$.

   Solution: Let $J = \{f(x) = a_n x^n + ... + a_1 x + a_0 \in \mathbb{Z}[x] : 5 \mid a_0\}$. Note that $\mathbb{Z}[x]$ is a commutative ring, so to show J is an ideal, it is sufficient to show that for all $f(x), g(x) \in J$, $f(x) - g(x) \in J$ and for all $h(x) \in \mathbb{Z}[x]$ and all $f(x) \in J$, $f(x)h(x) \in J$. Let $f(x), g(x) \in J$ be defined by $f(x) = a_n x^n + ... + a_1 x + a_0$ and $g(x) = b_m x^m + ... + b_1 x + b_0$. We have $5 \mid a_0$ and $5 \mid b_0$. Define $k = \max\{m, n\}$, $a_k = 0$ for all $k > n$ and $b_k = 0$ for all $k > m$. Then $f(x) - g(x) = (a_k - b_k)x^k + ... + (a_1 - b_1)x + a_0 - b_0$. Since $5 \mid (a_0 - b_0)$, $f(x) - g(x) \in J$. Now, take an arbitrary $h(x) \in \mathbb{Z}[x]$ and $f(x) \in J$. Assume $f(x) = a_n x^n + ... + a_1 x + a_0$ and $h(x) = c_m x^m + ... + c_1 x + c_0$. Then $h(x)f(x) = f(x)h(x) = a_n c_m x^{n+m} + (a_n c_{m-1} + a_{n-1} c_m)x^{n+m-1} + ... + a_0 c_0$. Since $5 \mid a_0$, then $5 \mid a_0 c_0$ and so $f(x)h(x) \in J$. Thus, we conclude that J is an ideal.

   For the second part, consider $f(x) = 5x^2 + 2x + 3$ and $g(x) = 5x^2 + x - 1$, both of which are elements of P, however, $f(x) - g(x) = x + 4$ is not an element of P, thus P is not a ring, and thus not an ideal.

2. Show that the set of non-units is an ideal in $\mathbb{Z}_8$.

   Solution: Let M denote the set of nonunits in $\mathbb{Z}_8$, i.e., $M = \{0, 2, 4, 6\}$. Note that for any $m, n \in M$, $m - n \in M$. Also, it is easy to show that $\forall k \in \mathbb{Z}_8$ and $\forall m \in M$, $mk = km \in M$. Thus, M is closed under subtraction and possesses the absorbtion condition, so we can conclude it is an ideal in $\mathbb{Z}_8$.

3. If I and J are ideals in a ring R, show that $I \cap J$ is an ideal in R. Is this the case for $I \cup J$?

4. Give an example of a subring in a given ring, which is not an ideal. Are there ideals which are not subrings?

5. If F is a field, R a nonzero ring, and $f : F \to R$ a surjective homomorphism, prove that f is an isomorphism.

6. Let $I = \{0, 5\}$ in $\mathbb{Z}_{10}$. Verify that I is an ideal. What are the elements in $\mathbb{Z}_{10}/I$? Show that $\mathbb{Z}_{10}/I \cong \mathbb{Z}_5$.

7. (a) Prove that the set T of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{R}$ is a subring of $M_2(\mathbb{R})$.
   (b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring T.
   (c) What are the cosets in T/I?
   (d) Prove that $T/I \cong \mathbb{R}$.

8. Let X be a rigid rhombus in the plane, and $G = Sym(X)$ its symmetry group (consisting of rotations and reflections).
   (a) List the elements of G. Name each by a letter and sketch the symmetry it represents.
   (b) Construct the operation table for G. Is G an abelian group?
   (c) List all the subgroups H of G.

9. Let G be the set of ordered triples of integers (a, b, c) with the following operation

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab')$$

   (a) Show that G is a group under $*$.
   (b) Is G abelian?

10. Let $GL(2, \mathbb{R})$ denote the group of units in the ring $M_2(\mathbb{R})$ if 2×2 matrices with real coefficients. What is the identity element in the group $M_2(\mathbb{R})$? How about in $GL(2, \mathbb{R})$? What is the order of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $GL(2, \mathbb{R})$? What is the order of $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $M_2(\mathbb{R})$?

11. Prove that if G is a group, its identity element is unique.

12. Let H be a subgroup of a group G. If e_G is the identity element of G and e_H is the identity element of H, prove that $e_G = e_H$.

    Solution: Note that $he_H = h$ for all elements $h \in H$, so in particular we have $e_He_H = e_H$. Similarly, $ge_G = g$ for all elements $g \in G$, so in particular we have $e_He_G = e_H$. Thus, $e_He_H = e_He_G$. Since G is a group, every element has an inverse, thus we have

$$\begin{aligned} e_H^{-1}(e_He_H) &= e_H^{-1}(e_He_G) \\ (e_H^{-1}e_H)e_H &= (e_H^{-1}e_H)e_G \\ e_H &= e_G \end{aligned}$$

13. Let a and n be two integers, such that $n > 1$ and $\gcd(a, n) = 1$. Let $\bar{a}$ denote the congruence class of a modulo n. Prove that $\bar{a}$ generates all of $\mathbb{Z}_n$, i.e. $\langle \bar{a} \rangle = \mathbb{Z}_n$.

14. Prove that the additive group $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic.

    Solution: Recall that

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{([0]_2, [0]_4), ([0]_2, [1]_4), ([0]_2, [2]_4), ([0]_2, [3]_4), ([1]_2, [0]_4), ([1]_2, [1]_4), ([1]_2, [2]_4), ([1]_2, [3]_4)\}$$

    and the order of each element is either 1, 2 or 4. Thus, the cyclic group generated by each element is of order 1, 2, or 4. Thus, no single element can generate the whole group $\mathbb{Z}_2 \times \mathbb{Z}_4$.

15. List all cyclic subgroups of (i) S_3, (ii) of U_9, (iii) of $\mathbb{Z}_9$.

Solution: The cyclic subgroups of S_3 are as follows.

$$G_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\rangle$$

$$G_2 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\rangle$$

$$G_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\rangle$$

$$G_4 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\rangle$$

$$G_5 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\rangle$$

The cyclic subgroups of $U_9 = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$ are as follows.

$$G_1 = \langle \bar{1} \rangle = \{\bar{1}\}$$

$$G_2 = \langle \bar{2} \rangle = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}, \bar{7}, \bar{5}\} = U_9 = \langle \bar{5} \rangle$$

$$G_3 = \langle \bar{4} \rangle = \{\bar{1}, \bar{4}, \bar{7}\} = \langle \bar{7} \rangle$$

$$G_4 = \langle \bar{8} \rangle = \{\bar{1}, \bar{8}\}$$

The cyclic subgroups of $\mathbb{Z}_9 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$ are as follows.

$$G_1 = \langle \bar{0} \rangle = \{\bar{0}\}$$

$$G_2 = \langle \bar{1} \rangle = \mathbb{Z}_9 = \langle \bar{2} \rangle = \langle \bar{4} \rangle = \langle \bar{5} \rangle = \langle \bar{7} \rangle = \langle \bar{8} \rangle$$

$$G_3 = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}\} = \langle \bar{6} \rangle$$

16. Challenge: If $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all a, b in G, prove that G is abelian.