Math 310-001 — **Review for Exam 2** — **Fall, 2014**

I. Review homework problems.

II. Review quizzes.

III. Be able to prove short and straightforward theorems (e.g. see Problems 4 and 8 below).

Some practice problems for review

1. Which of the following functions are isomorphisms, which are homomorphisms of rings, and which are neither?
   (a) $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(n) = 3n$.
   (b) $f : \mathbb{Z}_6 \to \mathbb{Z}_6$, defined by $f(n) = 3n$.
   (c) $g : \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \to \mathbb{R}$, defined by $g\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = x$.
   (d) $H : \mathbb{Q}[x] \to \mathbb{Q}[x]$, defined by $H(f(x)) = f^2(x)$.
   (e) $S : \mathbb{Z}_3[x] \to \mathbb{Z}_3[x]$, defined by $S(f(x)) = f^3(x)$.
   (f) $D : \mathbb{R}[x] \to \mathbb{R}[x]$ is the derivative map.
2. Let $K = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$. Show that the function $f : K \to K$, defined by $f(a + b\sqrt{5}) = a - b\sqrt{5}$ is an isomorphism of rings.
3. If $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map.
4. Let $f : A \to B$ be a homomorphism of rings. Define $C = \{f(a) : a \in A\}$. Prove that C is a subring of B.
5. Show that the first ring is not isomorphic to the second.
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_9$
   (b) $\mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_{18}$
   (c) $\mathbb{Z}_2[x]/(x^2)$ and $\mathbb{Z}_4$
   (d) $\mathbb{Z}$ and $\{2x : x \in \mathbb{Z}\}$
6. Let F be a field, $c \in F \setminus \{0\}$ and $f(x) \in F[x]$. Show that $f(x)$ and $f(x) + c$ are relatively prime.
7. Determine if $x^4 + x^2 + 1$ is reducible in $\mathbb{Z}_2$.
8. Let F be a field and $f(x), g(x), p(x) \in F[x]$, with $p(x) \neq 0_F$. It is given that the relation on $F[x]$ defined by $f(x) \equiv g(x) \pmod{p(x)}$ is an equivalence relation, i.e., it satisfies reflexivity, symmetry and transitivity. Let $[f(x)]_p$ denote the equivalence class of $f(x)$.
   Assume $[f(x)]_p \cap [g(x)]_p \neq \emptyset$.
   (i) Prove that $f(x) \equiv g(x) \pmod{p(x)}$.
   (ii) Use the above to show that $[f(x)]_p \subseteq [g(x)]_p$.
9. List the elements in $\mathbb{Z}_2[x]/(x^2 + x + 1)$. Is this a field? Why or why not?
10. List the elements in $\mathbb{Z}_3[x]/(x^2 + x)$. Is this a field? Why or why not?
11. Explain why $x + 1$ is a unit in $\mathbb{Z}_5[x]/(x^2 + 2)$ and find its inverse.
12. Show that $\mathbb{Q}[x]/(x^2 - 5)$ is isomorphic to $K = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$. Show that $f(x) = x^2 - 5$ has a root in K.